International Journal of Applied Science and Engineering
2004. 2, 1:16-28

Modification of Goodman-Cowin theory and its Application to the Constitutive Models of Flowing Granular Materials

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Abstract: Goodman and Cowin proposed in 1972 a continuum theory of a dry cohesionless granular material in which the solid volume fraction ν is treated as an independent kinematical field for which an additional balance law of equilibrated force is postulated. In the derivation of said balance of equilibrated of force there exists some logical inconsistency and it results in the incorrect explanation of this balance equation and the incorrect balance of internal energy. It is demonstrated that the balance of equilibrated force can be modified by a simple dimensional analysis. The resulted modified Goodman-Cowin theory is then applied to investigate the constitutive models of flowing granular materials. A complete thermodynamic analysis based upon Müller-Liu approach is performed and the constitutive responses of a granular material in thermodynamic equilibrium are obtained. From the theoretical investigations it shows that the results are more general than those obtained from the original theory and for simple shearing flow problems the current theory can reproduce all results from previous works based on the original theory.

Keywords: Goodman-Cowin theory; entropy inequality; constitutive models; granular materials.

1. Introduction

Granular materials are collections of a large number of discrete solid particles with interstices filled with a fluid or a gas. In most flows involving granular materials, the interstitial fluid plays an insignificant role in the transportation of momentum and thus flows of such materials can be considered dispersed single phase rather than multi-phase flows. Detailed reviews of flows of granular materials have been presented by Duran [4], Hutter & Rajagopal [10], Savage [18] and Wang & Hutter [22].

Granular materials are discrete in nature. Many theories have been developed to describe the behaviour of flowing granular materials, and generally the methods which are adopted can be classified into three different classes: the molecular dynamics approach, the statistical mechanics approach and the continuum mechanics approach. In adopting the continuum mechanics approach the discrete nature of granular materials will be "smeared" and consequently, granular materials exhibit then microstructural effects on their macroscale, which is accounted for, in general, by adding an additional dynamical equation for the solid volume fraction ν. Different authors do not unanimously agree upon the form of this equation. Svendsen and Hutter [19] treated the solid volume fraction as an internal

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variable and writed an evolution equation balancing its time rate of change with its production. Wilmanski [24] on the other hand, using statistical arguments on the microscale demonstrates that the Svendsen-Hutter equation needed to be complemented by a flux term, thus arriving at a complete balance law. Goodman and Cowin [2, 6, 7, 8] were on the ground of the theories of elastic materials introducing a balance law of equilibrated force in which the second time derivative of ν were balanced with a flux and a production term, and in comparison with other theories based upon the continuum mechanics approach the inclusion of $\ddot{\nu}$ is the most significant point of their theory.

In deriving their reduced constitutive relations from a class of constitutive postulate Goodman and Cowin used the classical Coleman-Noll approach of thermodynamics, i.e., the linear momentum equation, the energy balance and the balance of equilibrated force have all arbitrary assignable external source terms, so that these balance laws would not affect the exploitation of the entropy inequality. Whereas such a procedure can be tolerated for the momentum and energy sources, it is physically utter nonsense for the balance law of equilibrated force. This is an internal law all by itself, and at least this law must influence the thermodynamics. Recently Wang & Hutter [22] adopted their theory and rederived the constitutive models of a flowing granular material by Müller-Liu approach with which all balance laws are treated as constraints via Lagrange multipliers in the exploitation of entropy inequality. It shows that the results are more general and will agree with Coleman-Noll approach if the Helmholtz free energy does not depend upon $\dot{\nu}$.

Based upon the theories of elastic media Cowin and Goodman [3] used a variational principle to derive the balance equation of equilibrated force. In the derivation there exists some logical inconsistency and it will be shown that this inconsistency can be removed by a simple dimensional analysis. The modified Goodman-Cowin theory, especially the modified balance of equilibrated force, will be applied to investigate the constitutive models of a granular material by using Müller-Liu approach. It shows that the obtained constitutive responses in thermodynamic equilibrium are more general and for simple shearing flow problems the current theory can reproduce all results obtained by Wang & Hutter [23]. Firstly the Goodman-Cowin theory will be outlined, and some detailed and important comments on it will be given. Sequentially the modification of Goodman-Cowin theory will be presented and the resulted modified Goodman-Cowin theory will be applied to investigate the constitutive responses, especially the constitutive responses in thermodynamic equilibrium, of a granular material. This paper is summarized and commented in section 6.

2. Outline of Goodman-Cowin theory

The necessary thermal and mechanical field variables are introduced as primitive quantities. Specifically, apart from the velocity $\mathbf{v}$, there exists a kinematic variable, the volume fraction or the volume distribution function ν, introduced by Goodman [8], that accounts for the distribution of the solid volume in a granular body. It is complemented by the true mass density of grains γ, the stress tensor $\mathbf{t}$, body force $\mathbf{b}$, specific internal energy e, heat flux $\mathbf{q}$ and heat supply r. In addition, to account for energy flux and energy supply associated with the time rate of change of ν, a higher order stress and body force were introduced by Goodman and Cowin [6, 8]. Such terms are expected since the volume fraction and the motion of a granular body are assumed to be kinematically independent. Accordingly, a balance equation for ν is proposed which is called the balance of equilibrated force and it contains as phenomenological parameters the equilibrated inertia α, the equilibrated stress vector $\mathbf{h}$, the equilibrated intrinsic body force f and the equilibrated external body force w. The distributed solid body must satisfy the balance laws of motion of continuum mechanics. Accordingly, the following filed equations must be satisfied:

- Balance of mass $$(\gamma\nu)^{\bullet} + \gamma\nu \operatorname{div} \mathbf{v} = 0, \tag{1}$$
- Balance of linear momentum $$\gamma\nu\, \dot{\mathbf{v}} = \operatorname{div} \mathbf{t} + \gamma\nu\, \mathbf{b}, \tag{2}$$
- Balance of angular momentum $$\mathbf{t} = \mathbf{t}^{T}, \tag{3}$$
- Balance of equilibrated force $$\gamma\nu\alpha\ddot{\nu} = \operatorname{div} \mathbf{h} + \gamma\nu(\mathrm{f} + \mathrm{w}), \tag{4}$$
- Balance of internal energy $$\gamma\nu\dot{e} = \mathbf{t}\cdot\mathbf{D} + \mathbf{h}\cdot \operatorname{grad}\dot{\nu} - \gamma\nu \mathrm{f}\dot{\nu} - \operatorname{div}\mathbf{q} + \gamma\nu r. \tag{5}$$

Here $(\bullet)^{\bullet}$ denotes the time rate of change of $(\bullet)$ and equals to $\partial(\bullet)/\partial t + \operatorname{grad}(\bullet)\cdot\mathbf{v}$. In the theory the bulk density of a granular body ρ is decomposed into the production of the true mass density of grains γ and the volume fraction ν, i.e., $\rho = \gamma\nu$, and it is noted that for granular materials with incompressible grains, i.e., γ = const., the bulk density can still vary via the variation of volume fraction ν. The balance Eqs. (1~3) are analogous to the classical balance equations of mass, linear momentum and angular momentum. The balance of equilibrated force is assumed in this simplest form (4) accordingly to Goodman and Cowin [2, 6] and its derivation is based upon a variational principle (see Cowin and Goodman [3]). The balance of internal energy (5) differs from the traditional statement by the occurrence of the power terms associated with $\dot{\nu}$. Eqs. (1~5) should be regarded as constraints via Lagrange multipliers in the exploitation of the entropy inequality, and the entropy inequality is then investigated to identify the constitutive responses of a granular material.

3. **Comments on Goodman-Cowin theory**

(1) The balance of equilibrated force is a self-equilibrated force system. This type of equation arises also in the theories of dislocation, liquid crystal and elastic media (see Green & Rivlin [9], Leslie [12], Mindlin [15] and Toupin [20, 21]). Goodman [8] has shown that the higher order stress of the theories for elastic media degenerates to an equilibrated stress related to a system of self-equilibrated forces resulting in either a center of compression or a center of dilatation, i.e., the stress vector $\mathbf{h}$ can be regarded as a "double force without moment" and results in a center of compression or dilatation, which can be understood as the influence of dislocation in elastic media.

(2) Eq. (4) is a balance equation for the internal variable ν and under the scope of continuum mechanics ν cannot have any external source of forces, since all events associated with a self-equilibrated force balance system happen "inside" a material point, thus w can be treated simply as zero.

(3) The stored energy function ϕ used in the derivation of (4) (see Cowin and Goodman [3]), corresponds to the free energy in thermodynamic equilibrium under isothermal condition (see Fang [5]).

(4) Eq. (4) is devoted to the evolution of volume fraction ν, not to the pore space, which equals to 1-ν. But physically it can be understood that the pore space generates either an expansion or a contraction force.

(5) In comparison with other equations of this type the most significant point of (4) is that the second time rate of change of ν is included. The use of $\ddot{\nu}$ is rather reasonable since from the conservation of mass (1), $\dot{\gamma}$ can be described by $\dot{\nu}$ if div**v** is known. It means $\dot{\gamma}/\gamma$ and $\dot{\nu}/\nu$ bear some "similar mathematical structures". If a balance equation is proposed only for $\dot{\nu}$, then in the ensuring thermodynamics analysis determining the constitutive relations the effects of $\dot{\gamma}$ and $\mathrm{grad}\gamma$ cannot be appropriately taken into account.

(6) As indicated by Passman et al. [16], with Goodman-Cowin theory, especially with Eq. (4), it is impossible to show the difference between two granular bodies with uniform distribution of grains but of different sizes.

(7) Eq. (4) is a balance of energy, not a balance of force, since the equilibrated external body force w has the dimension of energy per unit mass (see Cowin and Goodman [3]), and its name "balance of equilibrated force" will result in incorrect interpretation. The left hand side of Eq. (4) emerges from the balance of linear momentum and its original form reads $d(\rho\alpha\dot{\nu})/dt$, consequently α has the dimension length. However, in view of Eq. (4) the dimension of α should be a length square; therefore there exists a logical inconsistency in the original derivation of (4).

Besides, according to Goodman and Cowin's assumption [6], the variation of ν is kinematically independent of the motion of the body; consequently the variation of ν provides an extra energy supply and energy flux under which the traditional balance of internal energy should be modified. In doing so, the kinetic energy, the energy flux and the energy supply associated with the variation of ν are assumed to be (the first column of (5)).

$$
\begin{array}{llll}
\frac{1}{2}\rho\alpha\dot{\nu}^2 & \text{kinetic energy associated with } \dot{\nu}, & \frac{1}{2}\rho|\mathbf{v}|^2 & \text{kinetic energy associated with } \mathbf{v}, \\
\dot{\nu}\mathbf{h} & \text{energy flux}, & \mathbf{vt} & \text{energy flux}, \\
\rho f\dot{\nu} & \text{energy supply.} & \rho\mathbf{b}\cdot\mathbf{v} & \text{energy supply.}
\end{array}
\tag{5}
$$

The postulation of the first column of (5) is based upon the similarity of the balance of linear momentum (the second column of (5)). However, the dimension of $\dot{\nu}$ is not vecloity, consequently the postulated kinetic energy, energy flux and energy supply associated with $\dot{\nu}$ should be modified. These all are the starting points of our modification of Goodman-Cowin theory.

4. Modification of Goodman-Cowin theory

We assume there exists a length scale ℓ associated with the volume fraction ν such that the time rate of change of the "momentum associated with ν" of a granular body B is postulated as

$$\frac{d}{dt}\int_B (\gamma\nu\,\ell\,\dot{\nu})\,dV\,. \tag{6}$$

Expression (6) should be balanced by its flux, supply and productions terms, viz.,

$$\frac{d}{dt}\int_B(\gamma\nu\ell\dot{\nu})dV=\int_{\partial B}\mathbf{h}\cdot\mathbf{n}dA+\int_B\gamma\nu(f+w)dV, \tag{7}$$

here ∂B denotes the surface of the granular body B and $\mathbf{n}$ is its outward unit vector. By using the conservation of mass, Reynolds transport theorem and the divergence theorem expression (7) can be transformed into

$$\int_B\left\{\gamma\nu(\dot{\ell}\dot{\nu}+\ell\ddot{\nu})-\mathrm{div}\mathbf{h}-\gamma\nu(f+w)\right\}dV=0. \tag{8}$$

Since dV is arbitrary chosen, in order to fulfill (8), the integrand of (8) should be zero and it leads to an evolution equation of ν in differential form

$$\gamma\nu(\dot{\ell}\dot{\nu}+\ell\ddot{\nu})=\mathrm{div}\mathbf{h}+\gamma\nu(f+w), \tag{9}$$

which is called here *the modified balance of equilibrated force* and becomes now a force balance, not an energy balance. Even though (9) is somewhat similar to (4), its physical meaning is complete different. Due to the force nature of (9) the following expressions are proposed for the energies associated with the variation of ν

$$\frac{1}{2}\rho(\ell\dot{\nu})^2:\text{kinetic energy associated with }\nu;\quad \ell\dot{\nu}\mathbf{h}:\text{energy flux};\quad \rho f\ell\dot{\nu}:\text{energy supply}, \tag{10}$$

since $\ell\dot{\nu}$ now has the dimension of velocity. With (10) the balance of internal energy is then modified as

$$\rho\dot{e}=\mathbf{t}\cdot\mathbf{D}-\mathrm{div}\mathbf{q}+\rho r+\mathbf{h}\cdot\mathrm{grad}(\ell\dot{\nu})-\rho f(\ell\dot{\nu}), \tag{11}$$

which differs from (5) apparently, and it is noted that in (11) the influence of ℓ is included. Eqs. (1~3, 9, 11) construct the *modified Goodman-Cowin theory*. There exist three different cases that the free length scale ℓ can behave:

- Case I : ℓ is regarded a constant,
- *Case II* : ℓ is regarded a constitutive quantity,
- *Case III* : ℓ is regarded an independent field quantity.

In the current paper only Case I will be discussed. The other two cases will be discussed in separated papers.

5. Application of modified Goodman-Cowin theory-Constitutive model of a granular material

5.1. Balance equations

The balance equations of the modified Goodman-Cowin theory are given by

- Balance of mass $\quad \dot{\gamma}\nu+\gamma\dot{\nu}+\gamma\nu\,div\mathbf{v}=0, \quad$ (12)
- Balance of linear momentum $\quad \gamma\nu\dot{\mathbf{v}}=div\,\mathbf{t}+\gamma\nu\mathbf{b}, \quad$ (13)
- Modified balance of equilibrated force $\quad \gamma\nu\ell\,\ddot{\nu}=div\,\mathbf{h}+\gamma\nu f, \quad$ (14)
- Balance of internal energy $\quad \gamma\nu\dot{e}=\mathbf{t}\cdot\mathbf{D}-div\,\mathbf{q}+\gamma\nu r+(\mathbf{h}\cdot grad\,\dot{\nu})\ell-\gamma\nu f\ell\,\dot{\nu}, \quad$ (15)

Eqs. (12~13) are identical with (1) and (2), respectively. With ℓ = const. the modified balance of equilibrated force (9) and the balance of internal energy (11) reduce to Eqs. (14~15). In Eq. (14) the equilibrated external body force w is not included since ν is an internal variable. The balance of angular momentum is not considered here because this requirement can be directly achieved in the assumption of the constitutive class of a granular body. Balance Eqs. (12~15) should be considered as constraints via Lagrange multipliers in the exploitation of the entropy inequality in the next subsection.

5.2. Entropy inequality

There is an additive quantity, the entropy, with specific density η, flux $\mathbf{\Phi}$, supply s and production π, for which we may write an equation of balance in the form

$$\pi = \gamma\nu\dot{\eta} + \operatorname{div}\mathbf{\Phi} - \gamma\nu s. \tag{16}$$

The entropy principle states that the entropy production π is non-negative in all thermodynamic processes, and so the entropy inequality must hold:

$$\gamma\nu\dot{\eta} + \operatorname{div}\mathbf{\Phi} - \gamma\nu s = \pi \geq 0. \tag{17}$$

Any process, which satisfies (17), represents a so-called admissible process. Such a process, however, must in addition satisfy the balance Eqs. (12~15). Liu [13,14] has shown that one can account for these balance Eqs. (12~15) in the entropy inequality (17) by employing Lagrange multipliers as follows

$$\begin{aligned}\pi = {} & \gamma\nu\dot{\eta} + \operatorname{div}\mathbf{\Phi} - \gamma\nu s - \lambda^{\gamma}(\dot{\gamma}\nu + \gamma\dot{\nu} + \gamma\nu \operatorname{div}\mathbf{v}) - \boldsymbol{\lambda}^{v}\cdot(\gamma\nu\dot{\mathbf{v}} - \operatorname{div}\mathbf{t} - \gamma\nu\mathbf{b}) \\ & - \lambda^{\nu}(\gamma\nu\ell\ddot{\nu} - \operatorname{div}\mathbf{h} - \gamma\nu f) - \lambda^{e}(\gamma\nu\dot{e} - \mathbf{t}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} - \gamma\nu r - (\mathbf{h}\cdot\operatorname{grad}\dot{\nu})\ell + \gamma\nu f\ell\dot{\nu}) \geq 0,\end{aligned} \tag{18}$$

and satisfying this new inequality for all (unrestricted) fields. Explicitly, the balance equations appear as constraints on the class of physically-realizable processes, where λ^{γ}, $\boldsymbol{\lambda}^{v}$, λ^{ν} and λ^{e} represent the corresponding Lagrange multipliers.

Introducing the free energy

$$\Psi = e - \theta\eta, \tag{19}$$

and introducing the assumption $\lambda^{e} = 1/\theta$ [a], where θ is the empirical temperature, and substituting them into (18) yields for the entropy inequality in the form

$$\begin{aligned}\pi = {} & \frac{\gamma\nu}{\theta}(\dot{e} - \dot{\theta}\eta - \dot{\Psi}) + \operatorname{div}\mathbf{\Phi} - \gamma\nu s - \lambda^{\gamma}(\dot{\gamma}\nu + \gamma\dot{\nu} + \gamma\nu \operatorname{div}\mathbf{v}) - \boldsymbol{\lambda}^{v}\cdot(\gamma\nu\dot{\mathbf{v}} - \operatorname{div}\mathbf{t} - \gamma\nu\mathbf{b}) \\ & - \lambda^{\nu}(\gamma\nu\ell\ddot{\nu} - \operatorname{div}\mathbf{h} - \gamma\nu f) - \frac{1}{\theta}(\gamma\nu\dot{e} - \mathbf{t}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} - \gamma\nu r - (\mathbf{h}\cdot\operatorname{grad}\dot{\nu})\ell + \gamma\nu f\ell\dot{\nu}) \geq 0.\end{aligned} \tag{20}$$

In deducing it, we assume that the material behaviour is independent of the supplies; so the sum of all external source terms in (20) must vanish, implying that

$$-\gamma\nu s + \boldsymbol{\lambda}^{v}\cdot\gamma\nu\mathbf{b} + \frac{\gamma\nu}{\theta}r = 0, \tag{21}$$

which serves as an equation determining the entropy supply in terms of other supply terms and is more general than the classical selection via the contribution of **b**. The entropy and its flux as well as the Lagrange multipliers must be considered as auxiliary quantities. This form of the entropy inequality (20) will be used to investigate the constitutive postulates in the next subsection.

5.3. Constitutive assumptions and restrict-Tions

We adopt Wang and Hutter's [23] postulation of the constitutive responses of a granular body in the form

$$C = \hat{C}(\nu_0, \nu, grad\,\nu, \dot{\nu}, \gamma, \theta, grad\ \theta, \mathbf{v}, \mathrm{grad}\ \mathbf{v}) \tag{22}$$

for the material quantities

$$\mathrm{C} = \{\Psi, \eta, \mathbf{t}, \mathbf{h}, \mathrm{f}, \mathbf{q}, \mathbf{\Phi}\}. \tag{23}$$

According to the assumptions of Goodman and Cowin [6], we assumed that the response functions for granular materials depend upon a reference configuration through the reference volume fraction ν_0. Invoking the principle of material objectivity the expression (22) reduces to

$$\mathrm{C} = \hat{\mathrm{C}}(\nu_0, \nu, \mathrm{grad}\nu, \dot{\nu}, \gamma, \theta, \mathrm{grad}\theta, \mathbf{D}), \tag{24}$$

where $\mathbf{D}$ denotes the symmetric part of the velocity gradient and is known as the stretching tensor. If the functional dependence of Ψ, $\mathbf{t}$, $\mathbf{h}$, $\mathbf{q}$ and $\mathbf{\Phi}$ in (24) is incorporated into the entropy inequality (20) by use of the chain rule of differentiation and using the identity

$$(\mathrm{grad}\,\nu)^{\cdot} = \mathrm{grad}\,\dot{\nu} - (\mathrm{grad}\,\nu)(\mathrm{grad}\,\mathbf{v}), \tag{25}$$

then the entropy inequality (20) becomes

$$\begin{aligned}
\pi = {} & \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\nu} - \lambda^{\gamma}\gamma - \frac{\gamma\nu}{\theta}f\ell\Big\}\dot{\nu} + \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\dot{\nu}} - \lambda^{\nu}\gamma\nu\ell\Big\}\ddot{\nu} + \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\gamma} - \lambda^{\gamma}\nu\Big\}\dot{\gamma} + \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\theta} - \frac{\gamma\nu}{\theta}\eta\Big\}\dot{\theta} \\
& + \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial grad\,\theta}\Big\}\cdot(grad\,\theta)^{\bullet} + \Big\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\mathbf{D}}\Big\}\cdot\dot{\mathbf{D}} + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\nu_0} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial\nu_0} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu_0} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu_0}\Big\}\cdot(grad\,\nu_0) \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\nu} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial\nu} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu}\Big\}\cdot(grad\,\nu) + \Big\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial grad\,\nu}\otimes grad\,\nu\Big\}\cdot grad\,\mathbf{v} \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\dot{\nu}} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial\dot{\nu}} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\dot{\nu}} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\dot{\nu}} + \frac{1}{\theta}\mathbf{h}\ell - \frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial grad\,\nu}\Big\}\cdot(grad\,\dot{\nu}) \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial grad\,\nu} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial grad\,\nu} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial grad\,\nu} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial grad\,\nu}\Big\}\cdot grad\,(grad\,\nu) \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\gamma} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial\gamma} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\gamma} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\gamma}\Big\}\cdot(grad\,\gamma) + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\theta} + \lambda^{\nu}\cdot\frac{\partial\mathbf{t}}{\partial\theta} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\theta} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\theta}\Big\}\cdot(grad\,\theta) \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial grad\,\theta} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial grad\,\theta} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial grad\,\theta} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial grad\,\theta}\Big\}\cdot grad\,(grad\,\theta) \\
& + \Big\{\frac{\partial\mathbf{\Phi}}{\partial\mathbf{D}} + \boldsymbol{\lambda}^{\mathrm{v}}\cdot\frac{\partial\mathbf{t}}{\partial\mathbf{D}} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\mathbf{D}} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\mathbf{D}}\Big\}\cdot(grad\,\mathbf{D}) - \lambda^{\gamma}\gamma\nu div\,\mathbf{v} + \lambda^{\nu}\gamma\nu f + \frac{1}{\theta}\mathbf{t}\cdot\mathbf{D} - \boldsymbol{\lambda}^{\mathrm{v}}\cdot\gamma\nu\dot{\mathbf{v}} \geq 0.
\end{aligned} \tag{26}$$

Let $\mathbf{x}$ be given by $\mathbf{x} = \{\dot{\mathbf{v}}, \ddot{\nu}, \dot{\gamma}, \dot{\theta}, (\mathrm{grad}\theta)^{\cdot}, \dot{\mathbf{D}}, \mathrm{grad}\nu_0, \mathrm{grad}\dot{\nu}, \mathrm{grad}(\mathrm{grad}\nu), \mathrm{grad}\gamma, \mathrm{grad}(\mathrm{grad}\theta), \mathrm{grad}\mathbf{D}\}$. It is now straightforward to see that the inequality (26) has the form

$$\mathbf{a}\cdot\mathbf{x} + \mathrm{b} \geq 0, \tag{27}$$

where the vector $\mathbf{a}$ and the scalar b are functions of the variables listed in (24), but not of $\mathbf{x}$, and the vector $\mathbf{x}$ depends on time and space derivatives of these quantities. Accordingly (27) is linear in $\mathbf{x}$, and since these variables can take any values, it would be possible to violate (27) unless

$$\mathbf{a} = \mathbf{0} \qquad \text{and} \qquad \mathrm{b} \geq 0, \tag{28}$$

where $(28)_1$ leads to the so-called Liu identities and $(28)_2$ gives rise to the residual entropy inequality. Explicitly, the entropy inequality must hold for all independent variations of $\mathbf{x}$. These variables appear linearly in (26), and thus their coefficients must vanish. It then follows that the Lagrange multipliers $\boldsymbol{\lambda}^{\mathrm{v}}$, λ^{ν} and λ^{γ} are given by

$$\boldsymbol{\lambda}^{\mathrm{v}} = \mathbf{0}, \quad \lambda^{\nu} = -\frac{1}{\ell\theta}\frac{\partial\Psi}{\partial\dot{\nu}}, \quad \lambda^{\gamma} = -\frac{\gamma}{\theta}\frac{\partial\Psi}{\partial\gamma}, \tag{29}$$

whilst the specific entropy density becomes

$$\eta = -\frac{\partial \Psi}{\partial \theta}. \tag{30}$$

Moreover, the free energy must obey the relations

$$\frac{\partial \Psi}{\partial \mathrm{grad}\theta} = \mathbf{0}, \quad \frac{\partial \Psi}{\partial \mathbf{D}} = \mathbf{0}, \tag{31}$$

and the equilibrated stress vector $\mathbf{h}$ (with $\lambda^{\mathrm{v}} = \mathbf{0}$) obtains

$$\mathbf{h} = \frac{\gamma\nu}{\ell}\frac{\partial \Psi}{\partial \mathrm{grad}\nu} - \frac{\theta}{\ell}\left\{\frac{\partial \mathbf{\Phi}}{\partial \dot{\nu}} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \dot{\nu}} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \dot{\nu}}\right\}, \tag{32}$$

whilst the relations (with $\lambda^{\mathrm{v}} = \mathbf{0}$)

$$\frac{\partial \mathbf{\Phi}}{\partial \nu_0} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \nu_0} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \nu_0} = \mathbf{0}, \quad \frac{\partial \mathbf{\Phi}}{\partial \gamma} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \gamma} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \gamma} = \mathbf{0}, \quad \frac{\partial \mathbf{\Phi}}{\partial \mathbf{D}} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \mathbf{D}} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \mathbf{D}} = \mathbf{0}, \tag{33}$$

$$\mathrm{sym}\left\{\frac{\partial \mathbf{\Phi}}{\partial \mathrm{grad}\,\nu} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \mathrm{grad}\,\nu} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \mathrm{grad}\,\nu}\right\} = \mathbf{0}, \tag{34}$$

$$\mathrm{sym}\left\{\frac{\partial \mathbf{\Phi}}{\partial \mathrm{grad}\,\theta} + \lambda^{\nu}\frac{\partial \mathbf{h}}{\partial \mathrm{grad}\,\theta} - \frac{1}{\theta}\frac{\partial \mathbf{q}}{\partial \mathrm{grad}\,\theta}\right\} = \mathbf{0}, \tag{35}$$

must hold among $\mathbf{\Phi}$, $\mathbf{h}$, $\mathbf{q}$, where $\mathrm{sym}\{\mathbf{A}\}$ denotes the symmetric part of a tensor $\mathbf{A}$. Eqs. (29~35) correspond to the condition $\mathbf{a} = \mathbf{0}$ in (28) and are known as Liu identities.

To simply the ensuring analysis, it is assumed here that the free energy is not a function of the time rate of change of ν. Under this assumption the emerging constitutive relations are in correspondence with those of Goodman and Cowin [2, 6, 7] which were gained by use of the classical Coleman-Noll approach. With $\Psi \neq \hat{\Psi}(\cdot, \dot{\nu})$, there follows

$$\lambda^{\nu} = -\frac{1}{\ell\theta}\frac{\partial \Psi}{\partial \dot{\nu}} = 0. \tag{36}$$

Furthermore, with this assumption and the condition (31) the functional dependence of Ψ reduces to

$$\Psi = \hat{\Psi}(\nu_0, \nu, \gamma, \theta, \mathrm{grad}\,\nu). \tag{37}$$

If we assume that (37) is isotropic with respect to $\mathrm{grad}\nu$, then it must have the form

$$\Psi = \hat{\Psi}(\nu_0, \nu, \gamma, \theta, \mathrm{grad}\,\nu \cdot \mathrm{grad}\,\nu), \tag{38}$$

from which immediately follows that the identity

$$\frac{\partial \Psi}{\partial \mathrm{grad}\nu} \otimes \mathrm{grad}\nu = \left[\frac{\partial \Psi}{\partial \mathrm{grad}\nu} \otimes \mathrm{grad}\nu\right]^{\mathrm{T}} \tag{39}$$

holds true, here ⊗ denotes dyadic product. In the next step we define the extra entropy flux vector $\mathbf{k}$ via the formula

$$\mathbf{\Phi} = \frac{\mathbf{q}}{\theta} - \boldsymbol{\lambda}^{\mathrm{v}}\mathbf{t} - \lambda^{\nu}\mathbf{h} + \mathbf{k}, \tag{40}$$

which, with $\boldsymbol{\lambda}^{\mathrm{v}} = \mathbf{0}$, $\lambda^{\nu} = 0$, reduces to

$$\mathbf{\Phi} = \frac{\mathbf{q}}{\theta} + \mathbf{k}. \tag{41}$$

Substituting (41) into (33)-(35) gives rise to the following identities for $\mathbf{k}$

$$\frac{\partial \mathbf{k}}{\partial \nu_0} = \mathbf{0}, \quad \frac{\partial \mathbf{k}}{\partial \gamma} = \mathbf{0}, \quad \frac{\partial \mathbf{k}}{\partial \mathbf{D}} = \mathbf{0}, \quad \mathrm{sym}\left\{\frac{\partial \mathbf{k}}{\partial \mathrm{grad}\nu}\right\} = \mathbf{0}, \quad \mathrm{sym}\left\{\frac{\partial \mathbf{k}}{\partial \mathrm{grad}\theta}\right\} = \mathbf{0}. \tag{42}$$

With condition (42) and under the requirement that **k** is an isotropic vector fucntion, **k** is then identified as

$$\mathbf{k} = \mathbf{0}. \tag{43}$$

Thus, the entropy flux takes its traditional form i.e., $\mathbf{\Phi} = \mathbf{q}/\theta$ is obtained. This result will not follow, when the Helmholtz free energy-depends upon $\dot{\nu}$. Indeed, in that case λ^{ν} is nontrivially determined by the free energy and so the entropy flux must deviate in direction from that of the heat flux by a contribution proportional to **h**, the equilibrated stress vector.

With (43) and $\lambda^{\nu} = 0$, expression (32) becomes

$$\mathbf{h} = \frac{\gamma\nu}{\ell}\frac{\partial\Psi}{\partial \mathrm{grad}\,\nu}. \tag{44}$$

In view of (38) and (44), another expression for the equilibrated stress vector **h** can be deduced as follows, namely

$$\begin{aligned}
\mathbf{h} &= \frac{\gamma\nu}{\ell}\frac{\partial\Psi}{\partial \mathrm{grad}\,\nu} = A\,\mathrm{grad}\,\nu,\\
A &= 2\frac{\gamma\nu}{\ell}\frac{\partial\Psi}{\partial(\mathrm{grad}\,\nu\cdot\mathrm{grad}\,\nu)} = \frac{\hat{A}(\nu_0,\nu,\gamma,\theta,\mathrm{grad}\,\nu\cdot\mathrm{grad}\,\nu)}{\ell}.
\end{aligned} \tag{45}$$

With this, the Liu-identities are now fully exploitated. Next we investigate the residual entropy inequality. The residual entropy inequality, corresponding to $(28)_2$, or $b \geq 0$, is given by

$$\begin{aligned}
\pi' = {} & \left\{-\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial\nu} - \lambda^{\gamma}\gamma - \frac{\gamma\nu}{\theta}f\ell\right\}\dot{\nu} + \left\{\frac{\partial\mathbf{\Phi}}{\partial\nu} + \boldsymbol{\lambda}^{v}\cdot\frac{\partial\mathbf{t}}{\partial\nu} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\nu} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\nu}\right\}\cdot(\mathrm{grad}\,\nu)\\
& + \left\{\frac{\gamma\nu}{\theta}\frac{\partial\Psi}{\partial \mathrm{grad}\,\nu}\otimes\right\}\cdot \mathrm{grad}\,\nu\,\mathrm{grad}\,\mathbf{v}\left\{\frac{\partial\mathbf{\Phi}}{\partial\theta} + \boldsymbol{\lambda}^{v}\cdot\frac{\partial\mathbf{t}}{\partial\theta} + \lambda^{\nu}\frac{\partial\mathbf{h}}{\partial\theta} - \frac{1}{\theta}\frac{\partial\mathbf{q}}{\partial\theta}\right\}\cdot(\mathrm{grad}\,\theta)\\
& - \lambda^{\gamma}\gamma\nu\,\mathrm{div}\,\mathbf{v} + \lambda^{\nu}\gamma\nu f + \frac{1}{\theta}\mathbf{t}\cdot\mathbf{D} \geq 0.
\end{aligned} \tag{46}$$

Substituting (29), (36), (38), (41), (43) and (44) into (46) results in the new form of the residual entropy inequality

$$\theta\pi' = (p - \beta - \gamma\nu f\ell)\dot{\nu} - \frac{\mathbf{q}\cdot\mathrm{grad}\,\theta}{\theta} + (\mathbf{t} + \nu p\mathbf{I} + \ell\mathbf{h}\otimes\mathrm{grad}\,\nu)\cdot\mathbf{D} \geq 0, \tag{47}$$

where the definitions

$$p = \gamma^2\frac{\partial\Psi}{\partial\gamma}\ :\ \text{thermodynamic pressure},\qquad \beta = \gamma\nu\frac{\partial\Psi}{\partial\nu}\ :\ \text{configuration pressure}, \tag{48}$$

have been introduced. At this point it should be pointed out that the proposed constitutive class (24) is only suitable for granular materials with compressible grains. For granular materials with incompressible grains, γ is no longer an independent filed quantity, and in this case one can delete γ from (24) and repeat the above thermodynamic analysis. It is found that the same constitutive restrictions can be obtained, provided that here the thermodynamic pressure p = $-\gamma\lambda^{\gamma}$ is no longer determined by the free energy as expressed in $(48)_1$, but becomes an unknown variable. The detailed thermodynamic analysis will not be repeated here.

5.4. Thermodynamic equilibrium

Further restrictions on the constitutive relations can be obtained from the residual entropy inequality (47) in the context of thermodynamic equilibrium, which is characterized in the current local formulation by vanishing the entropy production, i.e., $\theta\pi' = 0$. Inequality (47) can be represented as

$$\theta\pi' = \mathbf{X}(\mathbf{Y})\cdot\mathbf{Y} \geq 0,$$
$$\mathbf{X} = (p-\beta-\gamma\nu f\ell, \frac{-grad\theta}{\theta}, \mathbf{t}+\nu p\mathbf{I}+\ell\mathbf{h}\otimes grad\nu),$$
$$\mathbf{Y} = (\dot{\nu}, grad\theta, \mathbf{D}). \tag{49}$$

Thus, we define thermodynamic equilibrium as those states, in which the independent dynamic variables **Y** all vanish. Besides, $\theta\pi'$ also possesses its minimum value, namely 0, in thermodynamic equilibrium. Necessary conditions for this minimum are that

$$\left.\frac{\partial(\theta\pi')}{\partial Y_i}\right|_{\mathbf{Y}=0} = 0, \qquad Y_i \in \mathbf{Y}, \tag{50}$$

$$\left.\frac{\partial^2(\theta\pi')}{\partial Y_i \partial Y_j}\right|_{\mathbf{Y}=0} \text{ is positive semi-definite,} \quad Y_i, Y_j \in \mathbf{Y}. \tag{51}$$

(50) restricts the equilibrium forms of the dependent constitutive fields, while (51) constrains the signs of material functions in it. Here we will only deal with the first condition. It yields the equilibrium values of the equilibrated intrinsic body force f, the heat flux **q** and the stress **t**

$$f\big|_E = \frac{p-\beta}{\gamma\nu\ell}, \quad \mathbf{q}\big|_E = \mathbf{0}, \quad \mathbf{t}\big|_E = -\nu p\mathbf{I} - \ell\mathbf{h}\otimes grad\nu, \tag{52}$$

where the subindex E indicates that the indexed quantity is evaluated in thermodynamic equilibrium. Referring to (44) and (48) it is evident that the constitutive relations for the equilibrated intrinsic body force f, the equilibrated stress vector **h**, the heat flux **q** and the stress tensor **t** in thermodynamic equilibrium are known once the free energy Ψ is known. Expression $(52)_3$ also demonstrates that in equilibrium the stress need not to be a hydrostatic pressure. Clearly, **h** will play a significant role in the theory if the system makes inhomogenerous distribution of grains important. In that case, dilatant behaviour is observed and grains in close contact with each other can give rise to very high local stress. Furthermore, the existence of **h** gives rise to Mohr-Coulomb friction in equilibrium (see Cowin [1] and Savage & Jeffery [17]).

5.5. Remark

In the original theory Goodman and Cowin did not further discuss the role played by the equilibrated inertia α (here ℓ). The reason might be that, since in their formulation α appears only in the expression of the Lagrange multiplier λ^{ν}. If we assume that Ψ is not a function of $\dot{\nu}$, λ^{ν} will become zero and α will not enter the formulation of the constitutive relations for **h**, f, **q** and **t**. But referring to (44) and (52) it is apparent that in the current modified theory the free length scale ℓ does really enter the formulation in the constitutive relations for **h**, f, **q** and **t**. A comparison of the results obtained from the current and the original theories is summarized in Table 1.

6. Conclusions

In the current paper detailed discussions of Goodman-Cowin theory have been provided. It is pointed out that there exists some logical inconsistency in the original theory, and the inconsistency can be removed by a simple dimensional analysis. The most significant point of the Goodman-Cowin theory, namely, the balance of equilibrated force, has been logically corrected such that the resulted modified balance of equilibrated force becomes more physically reasonable. A complete thermodynamic analysis for the current modified Goodman-Cowin theory has also been performed. It shows that the current theory contains a free length scale ℓ and it does really enter the formulation in the constitutive relations for **h**, f, **q** and **t**, and accordingly has influence upon these constitutive quantities. Consequently, the current theory is more reasonable and general than the original one. The results also indicate some possible ways to postulate the non-equilibrium parts of **h**, f, **q** and **t**.

Table 1. A comparison between the current and original theories

	Original theory (Wang & Hutter's results [23]) α : equilibrated inertia; $[\alpha]=L^2$ $\gamma\nu\alpha\ddot{\nu} = \operatorname{div}\mathbf{h} + \gamma\nu f$,	**Current theory** ℓ : free length scale; $[\ell] = L$ $\gamma\nu(\dot{\ell}\dot{\nu} + \ell\ddot{\nu}) = \operatorname{div}\mathbf{h} + \gamma\nu f$, ℓ : constant
$\boldsymbol{\lambda}^{\mathbf{v}}$	$\boldsymbol{\lambda}^{\mathbf{v}} = \mathbf{0}$	$\boldsymbol{\lambda}^{\mathbf{v}} = \mathbf{0}$
λ^{ν}	$\lambda^{\nu} = -\dfrac{1}{\alpha\theta}\dfrac{\partial\Psi}{\partial\dot{\nu}}, \ \dfrac{\partial\Psi}{\partial\dot{\nu}} = 0$ (assumed)	$\lambda^{\nu} = -\dfrac{1}{\ell\theta}\dfrac{\partial\Psi}{\partial\dot{\nu}}, \ \dfrac{\partial\Psi}{\partial\dot{\nu}} = 0$ (assumed)
λ^{γ}	$\lambda^{\gamma} = \dfrac{-\gamma}{\theta}\dfrac{\partial\Psi}{\partial\gamma}$	$\lambda^{\gamma} = \dfrac{-\gamma}{\theta}\dfrac{\partial\Psi}{\partial\gamma}$
η	$\eta = -\dfrac{\partial\Psi}{\partial\theta}$	$\eta = -\dfrac{\partial\Psi}{\partial\theta}$
k	$\mathbf{k} = \mathbf{0}$ (deduced)	$\mathbf{k} = \mathbf{0}$ (deduced)
Ψ	$\Psi = \hat{\Psi}(\nu_0, \nu, \gamma, \theta, \operatorname{grad}\nu \cdot \operatorname{grad}\nu)$	$\Psi = \hat{\Psi}(\nu_0, \nu, \gamma, \theta, \operatorname{grad}\nu \cdot \operatorname{grad}\nu)$
h	$\mathbf{h} = \gamma\nu\dfrac{\partial\Psi}{\partial \operatorname{grad}\nu}$	$\mathbf{h} = \dfrac{\gamma\nu}{\ell}\dfrac{\partial\Psi}{\partial \operatorname{grad}\nu}$
$\mathbf{t}\vert_E$, $f\vert_E$, $\mathbf{q}\vert_E$	$\mathbf{t}\vert_E = -\nu p\mathbf{I} - \mathbf{h}\otimes\operatorname{grad}\nu$, $f\vert_E = \dfrac{p-\beta}{\gamma\nu}$, $\mathbf{q}\vert_E = \mathbf{0}$	$\mathbf{t}\vert_E = -\nu p\mathbf{I} - \ell\mathbf{h}\otimes\operatorname{grad}\nu$, $f\vert_E = \dfrac{p-\beta}{\gamma\nu\ell}$, $\mathbf{q}\vert_E = \mathbf{0}$
p, β	$p = \gamma^2\dfrac{\partial\Psi}{\partial\gamma}, \beta = \gamma\nu\dfrac{\partial\Psi}{\partial\nu}$	$p = \gamma^2\dfrac{\partial\Psi}{\partial\gamma}, \beta = \gamma\nu\dfrac{\partial\Psi}{\partial\nu}$

For simple shearing flow problems, non-equilibrium parts of **h**, f, **q** and **t** should be postulated such that the complete field equations can be obtained. If we adopt Wang and Hutter's [23] non-equilibrium parts of **h**, f, **q** and **t**, then the field equations obtained from the current theory are exactly the same as those from the original theory, which in turn means, that our theory can reproduce the all results obtained by Wang & Hutter [23]. From this point of view the current theory should not be understood to be non-useful, but should rather be regarded to be more general and reasonable than the original one.

As discussed before, there exist three different cases of the free length scale ℓ. In this paper only the first case, namely, ℓ is constant, has been discussed. The physical meaning of ℓ is still somewhat uncertain, but a bold interpretation can be a mean diameter of the grains or a length scale over which o localized volume fraction over which the influence of this grain is transmitted. The first case in which ℓ is considered a material constant is adequate for the first interpretation, while in the third case in which ℓ is considered an independent field quantity is valid for the second interpretation. The second case, in which ℓ is treated as a constitutive variable, is devoted to derive a more general functional dependence of this constitutive variable from which the theory developed by Passman et al. [16] can be generalized. Discussions for other two cases will be given in separated papers.

Acknowledgement

The author thanks the "DFG-Graduiertenkolleg" and Darmstadt University of Technology for the finical support, and Dr. Y. Wang of the Institute of Mechanics for his helpful discussions of the current paper.

Symbol list

b = external body force
c = constitutive class
d = symmetric part of velocity gradient
e = specific internal energy
f = equilibrated intrinsic body force
h = equilibrated stress vector
k = extra entropy flux vector
ℓ = free length scale
p = thermodynamic pressure
q = heat flux vector
r = specific external energy supply
s = specific external entropy supply
π' = residual entropy production
Ψ = helmholtz free energy
t = stress tensor
v = velocity
w = equilibrated external body force
α = equilibrated inertia
β = configuration pressure
ρ = bulk density
γ = true mass density of grains
ν = volume fraction
ν_0 = reference volume fraction
η = specific entropy density
θ = empirical temperature
π = entropy production
Φ = entropy flux vector
ϕ = stored energy function
λ^γ = lagrange multiplier corresponding to the balance of mass
λ^v = lagrange multiplier corresponding to the balance of linear momentum
λ^ν = lagrange multiplier corresponding to the balance of equilibrated force
λ^e = lagrange multiplier corresponding to the balance of internal energy

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