International Journal of Applied Science and Engineering
2004. 2, 3: 295-305

Limit Cycles in a General Nonlinear Oscillation

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Abstract: Theorems on the existence and uniqueness of limit cycles in the general nonlinear oscillation have been studied. The conditions that guarantee the uniqueness of limit cycles here are different from all the previous results. Several examples are given to illustrate that the theorems are easy to be employed, and they are useful in the discussion of limit cycles in quadratic differential equations and ecological systems.

Keywords: limit cycles; Liénard equation; nonlinear oscillation; ecological systems.

1. Introduction

Limit cycles of plane autonomous differential systems appeared in the very famous paper " Mémoire sur les courbes définies par une équation différentielle" of Poincaré (1881, 1886). In the 1930s, van der Pol and Andronov showed that the closed orbit in the phase plane of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After that, the existence, nonexistence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and scientists (see, for example, Ye et al. [17]).

The van der Pol equation

$$\frac{d^2x}{dt^2}+\varepsilon(x^2-1)\frac{dx}{dt}+x=0 \qquad (1)$$

can be generalized to the Liénard equation

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0. \qquad (2)$$

Let $G(x)=\int_0^x g(x)dx,\ F(x)=\int_0^x f(x)dx.$

By the Liénard transformation, the equation is equivalent to the following systems:

$$\frac{dx}{dt}=y-F(x),\ \frac{dy}{dt}=-g(x) \qquad (3)$$

or

$$\frac{dx}{dt}=-y-F(x),\ \frac{dy}{dt}=g(x), \qquad (4)$$

(see, for example, Arrowsmith and Place [1], or Ye et al. [17]).

The existence and uniqueness of limit cycles of the Liénard equation have been studied by many authors. There are, for

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example, Liénard [11], Dragiliëv [5], Filippov [6], Sanone [14], Levinson and Smith [10], Rytchkov [13], Zhang [19,20], and Zhon [21]. The Liénard systems (3) and (4) can be generalized to the following nonlinear oscillating systems:

$$\frac{dx}{dt} = h(y) - F(x), \quad \frac{dy}{dt} = -g(x) \tag{5}$$

and

$$\frac{dx}{dt} = -h(y) - F(x), \quad \frac{dy}{dt} = g(x). \tag{6}$$

Notice that, in general, the systems (5) and (6) are not equivalent to each other if $h(y)$ is not odd.

Zhang [19] obtained conditions for the uniqueness of limit cycles in the system (6) by the method of comparison. Later, Cherkas and Zhilevich [2] relaxed her conditions.

Zhang's theorem has been widely employed in the study of quadratic differential systems and ecological systems (see, for example, Ye et al. (1986), Zhang, et al. (1985), Kuang and Freedman (1988), Huang and Merrill (1989), Zhuo, et al. (1999), Liu, et al. (2000), Zheng, et al. (2001), Wo, et al. (2003), Dou, et al. (2003), Yan, et al. (2004), and Zhu (2004).

Since the existence of limit cycles in the system (6) has not been studied in Zhang [19,20], and Cherkas and Zhilevich [2], every time the theorem is used, the existence has to be discussed separately.

In this paper, we shall prove six theorems for the existence of limit cycles in the oscillating system (6), and discuss some new conditions that guarantee the uniqueness of limit cycles as well. Our results are different from those obtained by Zhang [19,20] and Cherkas and Zhilevich [2]. For example, the condition that $\frac{d}{dx}(\frac{F'(x)}{g(x)}) \geq 0$ is no longer needed.

Several examples are given to illustrate that our results can be easily employed in practice.

The general nonlinear oscillating systems are now often used in the studies of quadratic differential equations and ecological systems (see, Ye et al. [17], Huang [7], Kuang and Freedman [9], Huang and Merrill [8], Zhuo et al. [23], Liu et al. [12], Zheng et al. [18], Wo et al. [15], Dou et al. [4], Yan et al. [16], and Zhu [22]). However, in most of the recent papers [4,12,15,16,18,22,23], the authors just deal with a particular system itself without a general consideration of the theory of differential equations. So many similar proofs are repeating again and again for some particular systems. The system studied in this paper is quite general in mathematics, and many previous results can be easily derived by our theorems as special cases. And, of course, by using these theorems, one can produce new results in many different areas in addition to systems in the physical oscillations.

In our discussion, we assume that all the function in (6) are continuous and satisfy the uniqueness condition of solutions for $|x| < +\infty$ and $|y| < +\infty$.

We also assume that:

(A_1) $h(0) = 0$, $h(y)$ is increasing, $|h(\pm\infty)| = +\infty$; and $|y| < +\infty$ when $x \neq 0$, $xF(x) < 0$ for $|x|$ sufficiently small.

(A_2) $\overline{\lim}_{x\to\pm\infty} \left(G(x) + F(x)\operatorname{sgn} x\right) = +\infty.$

(A_3) a). $F(x)$ is bounded below for $x > 0$ if $\overline{\lim}_{x\to+\infty} F(x) < +\infty$;

b). $F(x)$ is bounded below for $x < 0$ if $\underline{\lim}_{x\to-\infty} F(x) > -\infty$.

Notice that, in this paper, $\overline{\lim}_{x\to a}$ stands for the upper limit as $x \to \pm\infty$, while $\underline{\lim}_{x\to\pm\infty}$, the lower limit as $x \to \pm\infty$.

2. Existence of limit cycles

In the following discussion, let $\Gamma: \{x = x(t),\ y = x(t) \,|\, x(t_0) = x_0,\ y(t_0) = y_0\}$ be the trajectory of the system (6) passing

(x_0, y_0) at $t = t_0$. We first prove

LEMMA 1. If (A_1) is satisfied, then

$$\overline{\lim}_{t\to+\infty} y(t) = +\infty \Rightarrow \overline{\lim}_{t\to+\infty} x(t) = +\infty, \quad \text{(i)}$$

$$\overline{\lim}_{t\to+\infty} y(t) = -\infty \Rightarrow \overline{\lim}_{t\to+\infty} x(t) = -\infty. \quad \text{(ii)}$$

PROOF: Assume that (i) is not true, then there is a solution for which

$$\overline{\lim}_{t\to+\infty} y(t) = +\infty, \text{ but } \overline{\lim}_{t\to+\infty} x(t) < +\infty. \quad (7)$$

Thus there exists an $a > 0$ such that $x(t) \le a$ for all $t > t_0$.

Let

$M_0 = \max_{0\le x\le a} |g(x)|$, and

$F_0 = \max_{0\le x\le a} |F(x)|$,

and h^{-1} be the inverse function of h.

Since $y(t)$ is increasing on the right half plane, we can choose t_1 and t_2, such that

$$x(t_1) > 0,\ y(t_1) = h^{-1}(2F_0),$$
$$x(t_2) > 0,\ y(t_2) > h^{-1}(2F_0) + \frac{M_0 a}{F_0}.$$

Consider the integral along the trajectory Γ, we have

$$y(t_2) = y(t_1) + \int_{x(t_1)}^{x(t_2)} \frac{g(x)}{-h(y) - F(x)} dx$$
$$\le h^{-1}(2F_0) + \left| \int_{x(t_1)}^{x(t_2)} \frac{|g(x)|}{|h(y)| - |F(x)|} dx \right| \quad (8)$$
$$\le h^{-1}(2F_0) + \frac{M_0 a}{F_0}.$$

This is a designed contradiction to $y(t_2) > h^{-1}(2F_0) + \dfrac{M_0 a}{F_0}$. Hence, (i) of Lemma 1 is valid. Similarly, we can prove (ii) of Lemma 1.

LEMMA 2. If (A_1), (A_2) are satisfied, and if (A_3 – a), then

$$\overline{\lim}_{t\to+\infty} x(t) = +\infty \Rightarrow \underline{\lim}_{t\to+\infty} y(t) = -\infty;$$

if (A_3 – b), then

$$\overline{\lim}_{t\to+\infty} x(t) = -\infty \Rightarrow \underline{\lim}_{t\to+\infty} y(t) = +\infty.$$

PROOF: we only prove (iii). Assume that there is a solution for which

$$\overline{\lim}_{t\to+\infty} x(t) = +\infty \text{ but } \overline{\lim}_{t\to+\infty} y(t) > -\infty, \quad (9)$$

Then, we can find a constant $b < 0$ such that $y(t) = \max\{h(b),\ b\}$ for all $t \ge t_0$. If $\overline{\lim}_{x\to+\infty} F(x) = +\infty$, then there exists an $x_1 > |x_0|$ such that $F(x_1) > -h(b)$. The continuity of $F(x)$ implies that there exists an $\varepsilon > 0$ such that $F(x) > -h(b)$ for $x \in (x_1 - \varepsilon, x_1 + \varepsilon)$.

Thus,

$$\frac{dx}{dt} = -h(y) - F(x) < 0,$$

which means that Γ can not cross the line $x = x_1 + \varepsilon$ if it enters the strip $|x - x_1| < \varepsilon$.

But it is impossible because of the hypothesis that $\overline{\lim}_{t\to+\infty} x(t) = +\infty$. Hence, (9) is incorrect.

If $\overline{\lim}_{t\to+\infty} F(t) < +\infty$, by $(A_3 - a)$, $F(x)$ is bounded below, and then there exists an f_0 (> 0) such that $|F(x)| < h^{-1}(f_0)$ for all $x > x_0$. The curve $-h(y) - F(x) = 0$ passes through the origin, and $x(t)$ increases monotonically above the curve. By the phase portrait analysis, the trajectory Γ either revolves around the origin or keeps traveling above with $x(t)$ increasing.

In both cases, we can find a piece of trajectory such that $x(t_1) = c(c > |x_0|,\ c = const.)$

$$-h(y(t)) - F(x(t)) \geq 0, \ (t_1 \leq t \leq t_2), \tag{10}$$

where $x(t_2)$ is so big that

$$\int_{x(t_1)}^{x(t_2)} g(x)dx > (-h(b) + h^{-1}(f_0))^2. \tag{11}$$

Now, the integral along Γ from t_1 to t_2 leads to

$$\begin{aligned} y(t_2) &= y(t_1) + \int_{x(t_1)}^{x(t_2)} \frac{g(x)}{-h(y) - F(x)} dx \\ &> h(b) + \int_{x(t_1)}^{x(t_2)} \frac{g(x)}{-h(b) + h^{-1}(f_0)} dx \\ &> h^{-1}(f_0). \end{aligned} \tag{12}$$

Since $h^{-1}(f_0) > h^{-1}(F(x))$ for $x > x_0$, then $y(t_2) > h^{-1}(-F(x(t_2)))$, which contradicts to (10). Therefore, (iii) is valid.

Similarly, we can prove (iv). From Lemmas 1 and 2, one can see that every trajectory of the system (6) revolves around the origin.

Thus, we have

LEMMA 3. If (A_1), (A_2) and (A_3-*a*) or (A_3-*b*) are satisfied, then that one of $x(t)$ and $y(t)$ is bounded implies the trajectory Γ is bounded.

Now, we are in the position to prove the existence theorems.

THEOTEM 4. If (A_1), (A_2), and (A_3 – a) are satisfied, and if there exists $M > -F(x)$, $(x > 0)$ such that

(i) $\displaystyle\int_0^{+\infty} \frac{g(x)}{M + F(x)} dx \leq \mu < +\infty$,

(ii) $h(y) \geq y$ for $y \geq M$;

then the system (6) has limit cycles.

PROOF: Consider the auxiliary system

$$\frac{dx}{dt} = h(y) + F(x), \quad \frac{dy}{dt} = -g(x) \tag{13}$$

and its trajectory

$$\Gamma^- : \{x = x(t), \ y = y(t) \big| x(0) = 0, \ y(0) = y_0, y_0 = M + C + \mu, \ C > 0\}.$$

We prove that

$$\lim_{\tau \to +\infty} x(\tau) = +\infty. \tag{14}$$

By (ii), when $y \geq M + C$,

$$h(y) + F(x) \geq C \quad (x \geq 0). \tag{15}$$

Clearly, when τ has a small increase from 0, $x(\tau)$ increase and $y(\tau)$ decrease. We claim that,

$$y(\tau) \geq M + C, \ \forall \tau > 0. \tag{16}$$

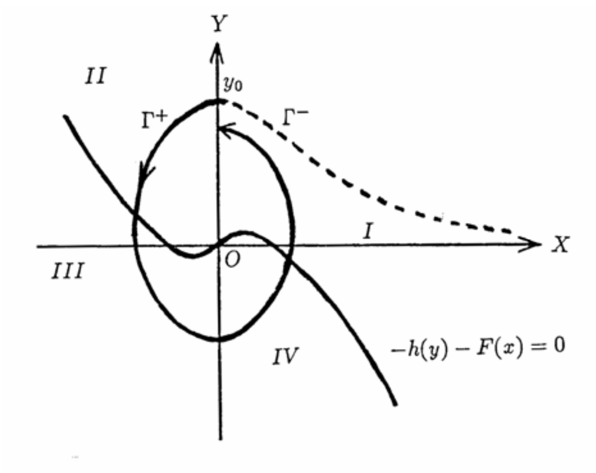

Figure 1. $\Gamma = \Gamma^+ \cap \Gamma^-$ and Γ^- is bounded above in y

Assume that (16) is not true. Then there exists τ' such that $y(\tau') < M + C$. By (i), and $h(y) + F(x) \geq 0$ for $y \geq M$,

$$y(\tau') = \int_0^{x(\tau')} \frac{-g(x)}{h(y)+F(x)}dx$$

$$\geq M + C + \mu - \int_0^{+\infty} \frac{g(x)}{M+F(x)}dx \qquad (17)$$

$$\geq M + C.$$

This contradiction shows that (16) is valid. Hence, (15) and then (14) are true. Since Γ^- is the negative semi-trajectory of τ of (6), Γ^- is bounded above in *y*. That is, $y(\tau) \leq y_0$ whenever $y(\tau) \in \Gamma^-$ (see Figure 1).

Let Γ^+ be the positive semi-trajectory of Γ. Then $y(t) \in \Gamma^+$ increases only when $x(t) > 0$. Since Γ^+ does not intersect with Γ^-, so $y(t)$ has a upper bound. By Lemma 3, Γ^+ is bounded. The origin (0, 0) is the only equilibrium of (6) and it is unstable. So the ω-limit set of Γ^+ is a limit cycle (see Cronin [3]). Similarly, we can prove

THEORIM 5. If $(A_1),(A_2)$ and (A_3 - *b*) are satisfied, and if there exists $M > F(X)$, $(x > 0)$ such that

(i) $\int_0^{-\infty} \frac{g(x)}{M-F(x)}dx \leq \mu < +\infty$;

(ii) $h(y) \leq y$ for $y \leq -M$.

then the system (6) has limit cycles.

Let us come back to the proof of Theorem 4. The conditions (i) and (ii) ensure that the trajectory Γ^- of the system (15) approaches to the positive infinity. Then Γ^+ can not cross the *y*-axis from left to right above y_0.

Therefore the outer boundary of the Bendixson region is granted. Actually, these conditions can be relaxed.

THEOREM 6. If (A_1), (A_2) and (A_3 - *a*), and if there exists *M* such that

$M > -F(x)$, $(x > 0)$, and

$\int_0^{+\infty} \frac{g(x)}{M+F(x)}dx < +\infty$;

then the system (6) has limit cycles.

PROOF: Since the origin $(0,0)$ is the only equilibrium that is unstable, we just need to prove that Γ $(t > 0)$ is bounded. By Lemma 3, it is equivalent to show that $y(t)$ has an upper bound. Otherwise, suppose $\overline{\lim}_{t\to+\infty} y(t) = +\infty$.

Since $y(t)$ only increases when $x > 0$, there exist t', $t'' > t_0$ such that $y(t')$ is sufficiently larger that $h(y) > M$ when y $> y(t')$, and

$$y(t'') > y(t') + \int_0^{+\infty} \frac{g(x)}{M+F(x)}dx,$$

for $x(t) > 0,\ t' \leq t \leq t''$

Because $y(t)$ increases while $x(t)$ decreases for $t \in [t', t'']$,

$$y(t'') = y(t') + \int_{x(t')}^{x(t'')} \frac{g(x)}{-h(y)-F(x)}dx$$

$$\leq y(t') + \int_{x(t'')}^{x(t')} \frac{g(x)}{M+F(x)}dx$$

$$\leq y(t') + \int_0^{+\infty} \frac{g(x)}{M+F(x)}dx.$$

The designed contradiction shows that $y(t)$ $(t > 0)$ has an upper bound. This completes the proof of Theorem 6.

Similarly, we have

THEOREM 7. If (A_1), (A_2) and (A_3 - *b*), and if there exists *M* such that $M > F(x)$, $(x < 0)$, and $\int_0^{-\infty} \frac{g(x)}{M-F(x)}dx < +\infty$, then the system (6) has limit cycles.

THEOREM 8. If (A_1), and (i), (ii) in Theorem 4 are satisfied, and if

(iii) there exists $x_1 < 0$ such that

$$-F(x_1) \geq h(M + C + \mu);$$

(iv) $\overline{\lim}_{x \to -\infty} F(x) = +\infty$, $F(x)$ is bounded below for $x > 0$, then the system (6) has limit cycles.

PROOF: Consider the positive and negative semi-trajectories Γ^+ and Γ^- in the proof of Theorem 4 (see Fig. 1). By (A_1), (i), (ii) and $F(x)$ is bounded below for $x > 0$,

$x(t) \in \Gamma^-$ goes to $+\infty$ as $t \to -\infty$.

That is, Γ^- has the same property as the one in the proof of Theorem 4. Thus, if we can show that Γ^+ comes back to the positive y-axis, the proof is done. As shown in Figure 1 the curve $-h(y) - F(x) = 0$ and the y-axis divide the xy-plane into four zones:

Zone I : $\{(x, y) | x > 0,\ -h(y) - F(x) < 0\}$,

Zone II : $\{(x, y) | x < 0,\ -h(y) - F(x) < 0\}$,

Zone III : $\{(x, y) | x < 0,\ -h(y) - F(x) > 0\}$,

Zone IV : $\{(x, y) | x > 0,\ -h(y) - F(x) > 0\}$.

When t increases, Γ^+ goes to Zone II in which both $x(t)$ and $y(t)$ decrease.

Let the line $x = x_1$ intersect the curve $-h(y) - F(x) = 0$ at (x_1, y_1). By (iii), $h(y_1) = -F(x_1) \geq h(M + C + \mu)$.

Therefore, $y_1 > y_0$ and Γ^+ must cross the curve $-h(y) - F(x) = 0$ into Zone III. In Zone III, $x(t)$ increases and $y(t)$ decreases, by the facts that $x(t)$ is bounded in Zone III. $(0,0)$ is unstable and

$$y(t) = y(t_0) + \int_{x(0)}^{x(t)} g(x(t))dt,\ \ y(t)$$

can neither approach to $-\infty$ nor to $(0,0)$.

It must cross the negative y-axis into Zone IV. The condition (iv) guarantees that Γ^+ enters Zone I, and finally comes back to the positive y-axis since it cannot touch Γ^-.

Similarly, we have

THEOREM 9. If (A_1) and (i), (ii) in Theorem 5 are satisfied, and if

(iii) there exists $x_2 > 0$ such that

$$-F(x_2) \leq h(-(M + C + \mu)),$$

(iv) $\underline{\lim}_{x \to -\infty} F(x) = -\infty$ $F(x)$ is bounded above,

(v) for all x < 0;then the system (6) has limit cycles.

3. Uniqueness of limit cycles

The proof of the uniqueness theorem of the limit cycles needs the following modified assumption:

(A_1)' $h(0) = 0$, $h(y)$ is increasing, $|h(\pm\infty)| = +\infty$; $xg(x) > 0$ when $x \neq 0$; and there exist $a < 0 < b$, N sufficiently large, such that x F(x) < 0 for $x \in (a, b), x \neq 0$, and $xF(x) < 0$, $F(x)$ is increasing for $x \in (-N, a)$ and $x \in (b, N)$.

Obviously, (A_1)' $\supset$ (A_1). That is, if (A_1)' is satisfied then so is (A_1). For conveience, we use the notation (A_1) instead of (A)' in the following discussion.

Let

$$G(x) = \int_0^x g(x)dx, H(y) = \int_0^y h(y)dy,$$

and

$$\lambda = G(x) + H(y).$$

THEOREM 10. If one of the following conditions is satisfied:

(i) $G(b) = G(a)$;

(ii) $G(b) > G(a)$ and there exist $x' \in (a, 0)$,

$y' < 0$ such that
$h(y') \ge -F(x'),\ H(y') \ge G(b)$;

(iii) $G(b) < G(a)$ and there exist $x'' \in (0,b)$, $y'' > 0$ such that
$h(y'') \le -F(x''),\ H(y'') \ge G(a)$;

(iv) there exist $x' \in (a,0)$, $y' < 0$ such that
$h(y') \ge -F(x'),\ H(y') \ge G(b)$; and
$x'' \in (0,b)$, $y'' > 0$ such that
$h(y'') \le -F(x''),\ H(y'') \ge G(a)$;

then in the system (6) there is at most one limit cycle.

PROOF: Let $x_1 = \min\{x : (x,y) \in \Gamma\}$, and $x_r = \max\{x : (x,y) \in \Gamma\}$, and Γ a limit cycle of (6). We first show that

CLAIM A.

$$x_1 < a < b < x_r, \tag{18}$$

that is, all the limit cycles contain the line segment $[a, b]$ on x-axis.

If not, assume that

CASE 1. $a < x_1 < x_r < b$.

Then differentiating λ along the system (6) results in

$$d\lambda = G'(x)dx + h(y)dy = -F(x)dy \ge 0,$$

since $F(x) > 0,\ dy < 0$ for $x \in (a,b)$, and $F(x) < 0,\ dy > 0$ for $x \in (0,b)$.

Notice that the equality is valid only when $x \ne 0$, we have

$$\oint_\Gamma d\lambda > 0. \tag{19}$$

This is impossible because Γ is a close curve.

CASE 2. $x_1 < a < x_r < b$.

In that case Γ crosses the positive x-axis,say, at $P(x_p, 0)$, $(0 < x_p < b)$, and the ray $x = a$, $y < 0$ at $P'(a, y_{p'})$, $(y_{p'} < 0)$.

Since $\int_{p'}^{p} d\lambda > 0$, $\lambda(P') < \lambda(P)$; that is

$$G(a) + H(y_{p'}) < G(x_p) < G(b). \tag{20}$$

By the fact that $H(y_{p'}) > 0$. (20) is impossible under the conditions (i) and (iii). We also can prove that Case 2 in not true under the trajectory Γ^- passing the point $B(b, 0)$.

When t decreases, Γ^- crosses the negative y-axis at $B'(0, y_{B'})$.

Since $d\lambda\Big|_{\widehat{BB'}} < 0$. $\lambda(B') < \lambda(B)$, then

$$H(y_{B'}) < G(b) \tag{21}$$

Moreover, consider the trajectory Γ^+ passing the point $C(x', y')$ (see Figure 2). When t increases, Γ^+ crosses the negative y-axis at $C'(0, y_{C'})$.

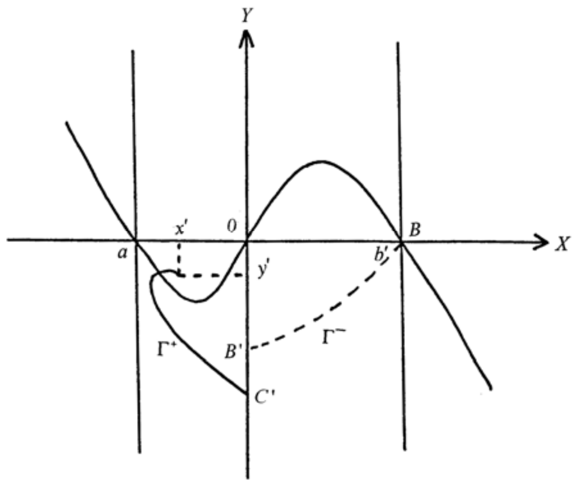

Figure 2. Γ^+ can not cross $\widehat{BB'}$

Since y is decreasing when $x < 0$, $y_{C'} < y'$. From (ii) or (iv) and (21),

$H(y_{C'}) > H(y') \geq G(b) > H(y_{B'})$,

which implies that $y_{C'} < y_{B'}$.

Because of the uniqueness of solutions, Γ^+ can not cross $\widehat{BB'}$ and it must intersect with the line $x = b$ as well. This designed contradiction completes the proof of Case 2.

CASE 3. $a < x_1 < b < x_r$.
In the case that Γ crosses the negative x-axis, say, at $Q(x_Q, 0)$, $a < x_Q < 0$, and the ray $x = b$, $y > 0$ at $Q'(b, y_{Q'})$, $y_{Q'} > 0$. Since $d\lambda_{\widehat{QQ'}} < 0$, $\lambda(Q') < \lambda(Q)$, and consequently,

$$G(b) + H(y_{Q'}) < G(x_Q) < G(a). \tag{22}$$

Since $H(y_{Q'}) > 0$, (22) is a contradiction to the conditions (i) and (ii). Now we consider the trajectory Γ^- passing $A(a, 0)$. When t decreases, Γ^- crosses the positive y-axis at $A'(0, y_{A'})$. Also, $d\lambda\big|_{\widehat{AA}} < 0$ implies that $\lambda(A') < \lambda(A)$. Hence

$$H(y_{A'}) < G(a). \tag{23}$$

The trajectory Γ^+ passing $D(x'', y'')$ will intersect with y-axis at $D'(0, y_{D'})$. Since y is increasing when $x > 0$, $y_{D'} > y''$. Therefore, by the conditions (iii) or (iv), and (23),

$$H(y_{D'}) > H(y'') \geq G(a) > H(y_{A'}). \tag{24}$$

Thus $y_{D'} > y_{A'}$. The uniqueness of solutions implies that Γ^+ can not cross $\widehat{AA'}$ and hence intersect with the line $x = a$. Consequently, Γ intersects with $x = a$ as well. This contradiction completes the proof of Case 3, and hence of Claim A.

We are now in a position to prove that the limit cycle is unique. If not, suppose there are two limit cycles Γ and Γ' and $\Gamma \subset \Gamma'$ (see Figure 3). Let us compute the integrals $\oint_{\Gamma} d\lambda$ and $\oint_{\Gamma'} d\lambda$.
As shown in Figure 3,

$\Gamma = \widehat{EF} \cup \widehat{FG} \cup \widehat{GH} \cup \widehat{HE}$, and
$\Gamma' = \widehat{E''F''} \cup \widehat{F''F'} \cup \widehat{F'G'} \cup \widehat{G'G''} \cup$
$\widehat{G''H''} \cup \widehat{H''H'} \cup \widehat{H'E'} \cup \widehat{E'E''}$

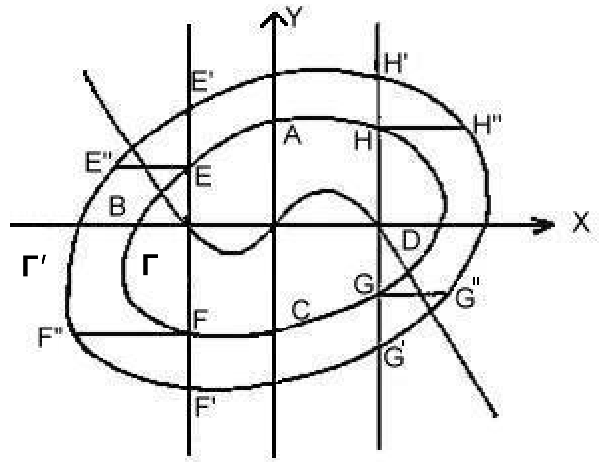

Figure 3. $\oint_{\Gamma'} d\lambda > \oint_{\Gamma} d\lambda$.

Since $\Gamma \subset \Gamma'$, for the same y, let

$(x_1, y) \in \Gamma$, $(x_2, y) \in \Gamma'$, we have

$$|x_1| < |x_2|. \tag{25}$$

By the fact that $F(x) > 0$, $dy < 0$, and $F(x)$ is decreasing for $x \in (-N, a)$, and (25),

$$\int_{\widehat{E''F''}} d\lambda = \int_{\widehat{E''F''}} -F(x)dy$$

$$> \int_{\widehat{EF}} -F(x)dy \tag{26}$$

$$= \int_{\widehat{EF}} d\lambda.$$

Similarly,

$$\int_{\widehat{G''H''}} d\lambda > \int_{\widehat{GH}} d\lambda. \tag{27}$$

Also, since $g(x)F(x) < 0$ for $x \neq 0$, $x \in (a,b)$, $dx < 0$, $-h(y) - F(x) < 0$, and the fact that $h(y)$ is increasing and for the same x, the y coordinate in Γ' is bigger than the one in Γ.

$$\int_{\widehat{H'E'}} d\lambda = \int_{\widehat{H'E'}} \frac{-F(x)g(x)}{-h(y)-F(x)} dx$$
$$> \int_{\widehat{HE}} \frac{-F(x)g(x)}{-h(y)-F(x)} dx \tag{28}$$
$$= \int_{\widehat{HE}} d\lambda.$$

Similarly,

$$\int_{\widehat{F'G'}} d\lambda > \int_{\widehat{FG}} d\lambda. \tag{29}$$

Considering the fact that integrals of $d\lambda$ along with $\widehat{E'E''}$, $\widehat{F''F'}$, $\widehat{G'G''}$, and $\widehat{H'H''}$ are all positive, we have

$$\int_{\Gamma'} d\lambda > \int_{\Gamma} d\lambda. \tag{30}$$

This is impossible because both $\oint_{\Gamma'} d\lambda$ and $\oint_{\Gamma} d\lambda$ are zeroes. This proves that there is at most one limit cycle in the system (6) if one of the conditions of Theorem 4 is satisfied. By the fact that (0, 0) is an unstable equilibrium, the limit cycle is stable if it exists. The proof of Theorem 10 is complete.

When $h(y) = y$, the system (6) is reduced to the Liénard system (3) or (4). The above results can be summarized as

THEOREM 11. If

(i) $xg(x) > 0$, $(x \neq 0)$; and there exist $a < 0 < b$, N sufficiently large, such that $xF(x) < 0$ for $x \in (a,b)$, $x \neq 0$, and $xF(x) < 0$, $F(x)$ is increasing for $x \in (-N,a)$ and $x \in (b,N)$;

(ii) one of the following holds

1). $\overline{\lim}_{x\to\pm\infty}\left(G(x) + F(x)\operatorname{sgn} x\right) = +\infty$;
$F(x)$ is bounded below for $x > 0$ if $\overline{\lim}_{x\to+\infty}\left(G(x) + F(x)\operatorname{sgn} x\right) < +\infty$; and there exists $M > -F(x)$, ($x > 0$) such that $\int_0^{+\infty} \frac{g(x)}{M + F(x)} dx < +\infty$;

2). $\overline{\lim}_{x\to\pm\infty}\left(G(x) + F(x)\operatorname{sgn} x\right) = +\infty$;
$F(x)$ is bounder above for $x < 0$ if $\overline{\lim}_{x\to+\infty}\left(G(x) + F(x)\operatorname{sgn} x\right) > -\infty$; and there exists $M > F(x)$, $(x > 0)$ such that $\int_0^{-\infty} \frac{g(x)}{M - F(x)} dx < +\infty$;

3). $\overline{\lim}_{x\to+\infty} F(x) = +\infty$; there exist $M > -F(x)$, $(x > 0)$, such that $\int_0^{+\infty} \frac{g(x)}{M + F(x)} dx < \mu < +\infty$ and $x_1 < 0$ such that $F(x_1) \leq -(M + C + \mu)$, $C > 0$;

4). $\overline{\lim}_{x\to-\infty} F(x) = -\infty$; there exist $M > F(x)$, (x < 0), such that $\int_0^{-\infty} \frac{g(x)}{M - F(x)} dx < \mu < +\infty$ and $x_2 > 0$ such that $F(x_2) \geq M + C + \mu$, $C > 0$;

(ii) one of the following holds

1). $G(b) = G(a)$;
2). $G(b) > G(a)$ and there exists $x' \in (a,0)$ such that $F(x') \geq \sqrt{2G(b)}$;
3). $G(b) < G(a)$ and there exists $x'' \in (o,b)$ such that $F(x'') \leq -\sqrt{2G(a)}$;
4). there exists $x' \in (a,0)$ and

$x'' \in (o,b)$ such that $F(x') \geq \sqrt{2G(b)}$ and $F(x'') \leq -\sqrt{2G(a)}$;

then the system (3) has a unique limit cycle.

4. Examples

Let us use some examples to illustrate our results.

EXAMPLE 1. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -y - F(x) \\ \frac{dy}{dt} &= 2x \end{aligned} \tag{31}$$

where

$$F(x) = \begin{cases} x^2(x-1) & if \quad x \geq 0, \\ x^4(x+1) & if \quad x < 0. \end{cases}$$

It is not difficult to see that the conditions (i), (ii)–1) (M = 1), and (iii)–1) (a = – 1, b = 1) in Theorem 11 are satisfied, and hence there is a unique limit cycle in (31).

However,

$$\frac{F'(x)}{g(x)} = \frac{5}{2}x^3 + 2x^2, \text{ if x < 0.} \tag{32}$$

Therefore $\frac{F'(x)}{g(x)}$ is decreasing if $-\frac{8}{15} < x < 0$, and hence Zhang's theorem [19], [20] and Cherkas and Zhilevich's theorem [2] are not applicable in the system (31).

EXAMPLE 2.

$$\begin{aligned} \frac{dx}{dy} &= -\frac{3y^2}{1+y^2} - x(x+\frac{1}{3})(x-1) \\ \frac{dy}{dt} &= \frac{2x}{1+x^4}. \end{aligned} \tag{33}$$

The system (33) has a unique limit cycle because the conditions of Theorem 6, and (ii) of Theorem 10, with

$$a = -\frac{1}{3},\ x' = -\frac{1}{4},\ b = y' = 1$$

are satisfied. But $G(\pm\infty) = \frac{\pi}{2} < +\infty$, and hence Zhang's theorem [19], [20] and Cherkas and Zhilevich's theorem [2] can not be employed either.

Acknowledgement

The author would like to thank Professor Lemin Zhu for her useful comments.

References

[1] Aarrowsmith, D. K. and C.M. Place, 1982. *"Differential Equations"*. Chapman and Hall, London.

[2] Cherkas, L. A. and L. I. Zhilevich, 1970. Some tests for the absence or uniqueness of limit cycles. *Differential Equations*, 6: 891-897.

[3] Cronin, J. 1980. *"Differential Equations"*. Marcel Dekker, New York.

[4] Dou, J. H. and W. Huang. 2003. Qualitative analysis of a nonlinear biochemical system. *Applied Mathematics, Journal of Chinese Universities Series A*, 18, 2: 158-162.

[5] Dragiliëv, A. V. 1952. Periodic solutions of the differential equation nonlinear oscillations. *Prikl. Mat. Mekh.* 16: 85-88 (in Russian).

[6] Filippov, A. F. 1952. A sufficient condition for the existence of a stable limit cycle for a second-order equation. *Mat. Sb.* 72, 30: 171- 180 (in Russian).

[7] Huang, Xun-Cheng. 1988. Uniqueness of limit cycles of generalized Liénard systems and predator-prey systems. *Journal of Physics A*: *Mathematics and Genera*, 21: L685 – 691.

[8] Huang, Xun-Cheng and Stephen Merrill, 1989. Conditions for uniqueness of limit

cycles in general predator-prey system. *Mathematical Biosciences* 96: 47-60.

[9] Kunag, Y. and H. I. Freedman. 1988. Uniqueness of limit cycles in Gauss-type model of predator-prey systems, *Mathematical Bioscience* 88: 67-84.

[10] Levinson, N. and O. K. Smith. 1942. A general equation for relaxation oscillations. *Duke Mathematics Journal*, 9: 382-403.

[11] Liénard A. 1928. *Etude des oscillations entretenuses*. Rev. Général de 1'Électricité, 23: 901-902.

[12] Liu, X. S. and Z. H. Zhao. 2000. Existence and uniqueness of limit cycle of cubic Kolmogorov differential system. *Journal of Biomathematics*, 15, 3: 266-271.

[13] Rytchkov, G. S. 1970. A complete investigation of the number of limit cycles of the equation $(b_{10}x+y)\ dy = \sum_{i+j=1}^{2} a_{ij}x^i y^j\ dx$, *Differential Equations*, 6: 1665 – 1670.

[14] Sansone, G., 1949. Sopra lequazione di A. Lienard delle oscillazioni di rilassimento, *Annali Di Matematicat Pura ED Applicatae*, 28, 4: 153-181.

[15] Wo, S. and Y. Zou, 2003. The analysis of plant eater-plant system with mutual interference. *Acta Mathematicae Applicatae Sinica*, 26, 4: 708-714.

[16] Yan X. and C. Zhang. 2004. The limit cycle of a kind of Kolmogorov's system. *Journal of Gansu Sciences*, 16, 1: 23-26.

[17] Ye, Y. Q., S. L. Wang and X. A.Yang. 1886. "*Theory of Limit Cycles*". American Mathematical Society, Providence, Rhode Island.

[18] Zheng, J. P., et al. 2001. Existence and uniqueness of limit cycle about prey-predator system with sparssing effect. *Journal. of Biomathematics*, 16, 2: 156-161.

[19] Zhang, Zhi-fen. 1986. Proof of the uniqueness Theorem of limit cycles of generalized Lienard equation. *Applicable Analysis*, 23: 67-76.

[20] Zhang Zhi-fen. T. R. Ting. W. Z. Huang and Z. X. Dong, 1985. "*Qualitative Theory of Differential Equations*". Science. Beijing, (in Chinese).

[21] Zhon Y., 1980. Existence of limit cycles in a nonlinear oscillation. *Acta Mathematicae Applicatae Sinica*, 3: 1, 50 – 56.

[22] Zhu, Q. 2004. The qualitative analysis of a sort of plane and space nonlinear autonomous systems. *Journal. of Basic Science and Engineering*, 12, 1: 1-12.

[23] Zhuo, X. and Y. Zhang. 1999. The qualitative analysis of a class of n+2 degree systems in plane. *Applied Mathematics, Journal of Chinese Universities Series A*, 14, 2: 130-135.