A New Recurrent Approach for Phase Unwrapping

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Abstract: True phase including phase characteristic of different complex functions and transfer functions is calculated by applying $\mathrm{modulo}\ 2\pi$ operation to tangent inverse function. Tangent inverse function is value limited to $-\pi/2 < x < \pi/2$. Different artificial discontinuities or phase jumps appear in the phase function. The problem of phase unwrapping is considered here and a new phase unwrapping method for removing artificial phase jumps was described and tested. A quantitative study is made to compare $\mathrm{modulo}\ 2\pi$ unwrapping algorithm with the proposed algorithm in this paper.

Keywords: Phase Unwrapping; Butterworth filter; SAR; $\mathrm{modulo}\ 2\pi$ operation; recurrent algorithm.

1. Introduction

Over the past several years there has been a marked increase in the application of coherent signal and image processing. Coherent processing requires an accurate estimate of the phase [1, 2]. Examples where coherent processing is required include synthetic aperture radar (SAR) [3, 4, 5, 6, 7], synthetic aperture sonar, adaptive beam-forming, acoustic imaging, projection and diffraction tomography, adaptive optics, inverse synthetic aperture radar (ISAR) [8], magnetic resonance imaging (MRI), field mapping, echo planar imaging [9, 10, 11, 12], chemical shift mapping [13, 14, 15] velocity measurements [16, 17], and others.

Unfortunately, one is only able to measure a wrapped version of the phase not the true phase. The measured phase $\phi(n)$ at instant n is actually obtained from the true phase, $\phi_T(n)$ by a modulo operation as follows: $\phi(n) = \phi_T(n)\ \mathrm{modulo}\ 2\pi$. This wrapping leads to artificial phase jumps being introduced near boundaries.

In many applications, the true phase relates to some physical quantity such as surface topography in interferometry [18, 19, 20], the degree of magnetic field inhomogeneity in the water/fat separation problem in MRI [14], The measured nonlinear phase does provide useful information. The phase, however, must be unwrapped before further use. The phase unwrapping problem is then to obtain an estimate for the true phase from the measured wrapped phase. Thus, the wrapped value must be unwrapped through some method to estimate true phase which is the quantity relating to the physical property of interest.

Many attempts were done to correct this

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problem [19, 21, 22] however the problem in general is not solved. Basically all the existing phase unwrapping techniques start from the fact that it is possible to estimate the neighboring pixel differences of the unwrapped phase when these differences are less than π. From these, the unwrapped phase can be reconstructed up to an additive constant. The methods differ in the way they overcome the difficulty posed by the fact that this hypothesis may be somewhere false, which cause the estimated unwrapped phase differences to be inconsistent, that is their "integral" depends on the integration path.

Existing unwrapping techniques can be categorized according to their: Dimensionality (1D, 2D, 3D, etc.); applications (SAR, general optical interferometry, MR angiography, MR chemical shift mapping or field mapping); and approach (fitting function, cost function optimization, filtering, region growing/ merging).

Methods based on fitting function (usually truncated Taylor series [23, 24] or polynomials [9, 13, 25] can be easily generalized to work with data of any dimension. However, in these methods functions can not vary too rapidly. This is undesirable for mapping applications where there are small, rapidly changing regions.

Branch cuts methods [20, 26] unwrap by integrating the estimated neighboring pixel differences of the unwrapped phase along paths avoiding the regions where these estimated differences are inconsistent. The problem of building cuts delimiting these regions is very difficult and the resulting phase unwrapping algorithm is very expensive computationally.

In least squares methods [27, 28, 29], unwrapping is achieved by minimizing the mean square deviation between the estimated and the unknown neighboring pixel differences of the unwrapped phase. Least squares methods are very efficient computationally when they make use of fast Fourier transform techniques [30, 31]. But the resulting unwrapping is not

very accurate, because least squares procedures tend to spread the errors that are instead concentrated on a limited set of points. To overcome this problem a weighting of the wrapped phase can be useful. However, the weighted least squares algorithms proposed [6, 30] are iterative and not as efficient as the unweighted ones. Moreover, the accuracy of the results depends on the weighting mask used.

In this paper a new method for phase unwrapping is proposed. Instead of using $modulo\ 2\pi$ operation for removing artificial phase jumps in phase characteristics, a recurrent formula is used to calculate the phase.

Calculation is started by calculating the phase of first point, and then the phase difference between the neighboring points is calculated and added to the previous phase until the phase at the desired point is calculated. Finally, tests of proposed method are presented. The results are really satisfactory: they demonstrate the consistency and efficiency of the method, which seems to be accurate. Comparative tests with $modulo\ 2\pi$ operation are presented too.

2. Problem formulation

The transfer function of any linear system could be defined as:

$$H(\omega) = R(\omega) + jI(\omega) = A(\omega)e^{j\Phi(\omega)} \qquad (1)$$

Where

$$A(\omega) = \sqrt{R^2(\omega) + I^2(\omega)} \qquad (2)$$

$$\Phi(\omega) = arctg\left(\frac{I(\omega)}{R(\omega)}\right) = \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right) \qquad (3)$$

$y = arctg(x)$ is the inverse function of $\tan(z)$ for values of z on the interval $-\pi/2 < z < \pi/2$ as shown in Figure 1. The function is monotonic increasing in the defi-

nition interval and has a value $y = 0$ at $x = 0$ and it has odd symmetry.

For many systems the phase characteristic is defined outside the definition interval of the arctg-function, which means that when calculating the phase characteristic using the arctg-function many artificial phase jumps being introduced near boundaries in the phase characteristic. For example all-pass filter has transfer function given by:

$$H(\omega) = A e^{-j\omega\tau} = A\left(\cos\omega\tau - j\sin\omega\tau\right) \quad (4)$$

where A and τ are constants.

For this example the mathematical expression for phase is known. In many cases the signal or the transfer function are given experimentally as a sequence of discrete values and the phase is calculated using Equation (3). It cannot be sure that the jumps in the phase characteristic calculated are natural in characteristic or they are inserted in it due to tangent inverse function. The phase and group delay correction depends on the calculation and does not depend on the algorithm used to calculate tangent inverse function.
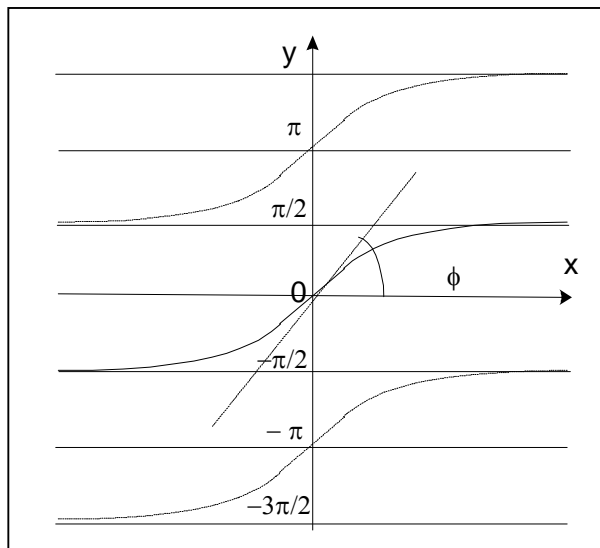

Figure 1. Plots of arctg-function

So the phase characteristic of this filter is given by:

$$\Phi(\omega) = arctg\left(\frac{I(\omega)}{R(\omega)}\right) = arctg\left(\frac{-\sin(\omega\tau)}{\cos(\omega\tau)}\right)$$

$$\Phi(\omega) = arctg\left[-\tan(\omega\tau)\right] \quad (5)$$

The actual phase of this filter is $(-\omega\tau)$ but the phase characteristic calculated using Equation (3) is not the same outside the values interval as shown in Figure 2.
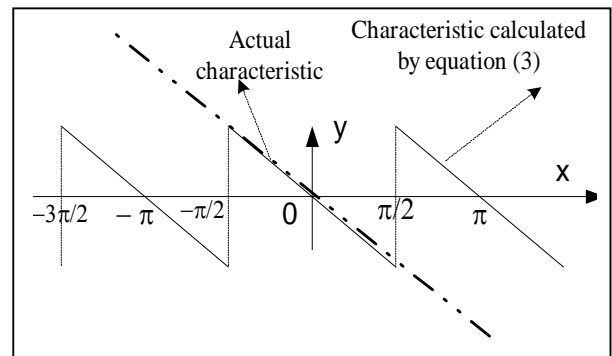

Figure 2. All-Pass fiter phase characteristic

3. The proposed algorithm

The tangent of the difference of two angles is given by:

$$\tan(\varphi_k - \varphi_{k-1}) = \frac{\sin(\varphi_k - \varphi_{k-1})}{\cos(\varphi_k - \varphi_{k-1})}$$

$$\tan(\varphi_k - \varphi_{k-1}) = \frac{\tan\varphi_k - \tan\varphi_{k-1}}{1 + \tan\varphi_k \tan\varphi_{k-1}} \quad (6)$$

But the tangent of angles of complex numbers are given by:

$$\tan(\varphi_k) = \frac{\text{Im}(\dot{A}_k)}{\text{Re}(\dot{A}_k)}$$

and

$$\tan(\varphi_{k-1}) = \frac{\text{Im}(\dot{A}_{k-1})}{\text{Re}(\dot{A}_{k-1})}$$

where $\dot{A}_k, \dot{A}_{k-1}$ are the values of the complex function at points: $k,$ and $(k-1)$ respectively.

By denoting:

$$x = \tan\left(\varphi_k\right) = \frac{\operatorname{Im}\left(\dot{A}_k\right)}{\operatorname{Re}\left(\dot{A}_k\right)} \qquad (7)$$

and

$$y = \tan\left(\varphi_{k-1}\right) = \frac{\operatorname{Im}\left(\dot{A}_{k-1}\right)}{\operatorname{Re}\left(\dot{A}_{k-1}\right)} \qquad (8)$$

The tangent of the difference of two angels is given by:

$$\tan\left(\varphi_k - \varphi_{k-1}\right) = \frac{x - y}{1 + xy} \qquad (9)$$

And the phase difference between any neighboring points is given by:

$$\Delta\varphi_k = \left(\varphi_k - \varphi_{k-1}\right) = arctg\left(\frac{x-y}{1+xy}\right) \qquad (10)$$

Because $arctg$ is determined in interval $\left(-\pi/2, \pi/2\right)$ and the actual phase difference can be in the interval $0 < \Delta\varphi_k < 2\pi$ the actual phase difference must be determined from the knowledge of the values of the complex number to determine the quadrant in which the phase difference lies so that the following cases can be realized:

i. $xy > -1$.

In this case $\Delta\varphi_k$ within interval $\left(\mp\pi/2\right)$ and $\Delta\varphi_k = arctg\left(\frac{x-y}{1+xy}\right)$

ii. $xy < -1$ and $x > 0$

In this case

$$\Delta\varphi_k = \pi + arctg\left(\frac{x-y}{1+xy}\right)$$

iii. $xy < -1$ and $x < 0$

In this case

$$\Delta\varphi_k = -\pi + arctg\left(\frac{x-y}{1+xy}\right)$$

iv. $xy = -1$ and $x > y$

In this case $\Delta\varphi_k = \pi/2$

v. $xy = -1$ and $x < y$

In this case $\Delta\varphi_k = -\pi/2$

To overcome the problem of phase discontinuity the following algorithm is proposed:

Suppose that the phase characteristic of a complex array Z, with N points to be calculated, then the algorithm works as follows:

(a) With assumption that the phase at the first point $|\varphi_0| \le \pi/2$, the initial phase of the first point in the array Z is calculated as:

$$\varphi_0 = arctg\left(\frac{\operatorname{Im}\left(\dot{A}_0\right)}{\operatorname{Re}\left(\dot{A}_0\right)}\right) \le \frac{\pi}{2}$$

The phase at the second point in the array Z is calculated as $\varphi_1 = \varphi_0 + \Delta\varphi_1$

(b) The phase difference between any point k and the previous point $(k-1)$ is calculated as $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$

(c) With condition that $\Delta\varphi_k \le \pi/2$ the phase difference $\Delta\varphi_k$ can be calculated as:

$$\Delta\varphi_k = \left(\varphi_k - \varphi_{k-1}\right) = arctg\left(\frac{x-y}{1+xy}\right)$$

(d) The phase at any point k in the array Z is calculated using the following recurrent equation:

$$\varphi_k = \varphi_{k-1} + \Delta\varphi_k = \varphi_0 + \sum_{i=1}^{k}\Delta\varphi_i \qquad (11)$$

A flow chart for the proposed algorithm is shown in Figure 3.

4. Examples

The first example will consider the calculation of a phase characteristic of an all-pass filter. Results of running a program that realizes this algorithm is provided in Figure 4. It

can be seen that the algorithm completely fixes the problem of phase jumps.

The second example consider a prototype low-pass fourth-order Butterworth filter with cut-off frequency of 1rad/sec. The transfer function of this filter is given by:

$$K(s) = \frac{1}{s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1}$$

The resultant graphics of phase characteristic achieved by running the proposed algorithm are plotted in Figure 5. From the plots of the phase characteristic it can be seen how the algorithm corrected the unnatural jumps and built a natural phase characteristic for the forth order filter.

5. Discussion

The UNWRAP-function algorithm removes phase discontinuities in array by adding or subtracting an appropriate multiple of 2π to each element. The function algorithm recognizes a phase discontinuity, or phase jump, whenever a given input element, u_i differs from the preceding element by an absolute amount greater than the absolute value of the specified tolerance parameter, (α). To eliminate the jumps in the input, the unwrap adds $2\pi k$ to all elements following a jump (including the discontinuous element itself). The value of k is initialized to 0, and is then incremented by 1 for each successive negative jump $\left[abs(u_i - u_{i-1}) > abs(\alpha); u_i < u_{i-1} \right]$, and decremented by 1 for each successive positive jump $\left[abs(u_i - u_{i-1}) > abs(\alpha); u_i > u_{i-1} \right]$. One possible realization of UN-WRAP-function flowchart is shown in Figure 6.

To compare the results of the proposed algorithm with the results of MATLAB function UNWRAP we shall consider a 25-Th order low-pass FIR filter design using the window method. The phase characteristic of this filter is plotted in Figure 7. From which it is

seen the correctness of the proposed algorithm and how it calculates the phase perfectly. The phase characteristic calculated by UNWRAP still has discontinuity.
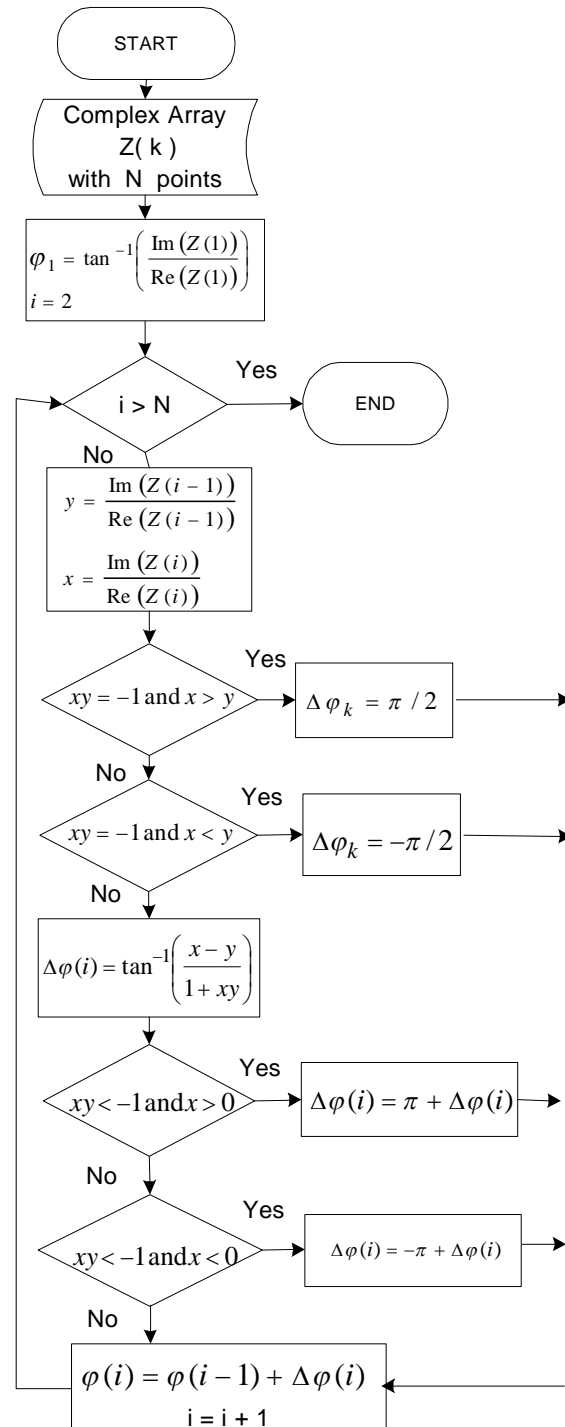

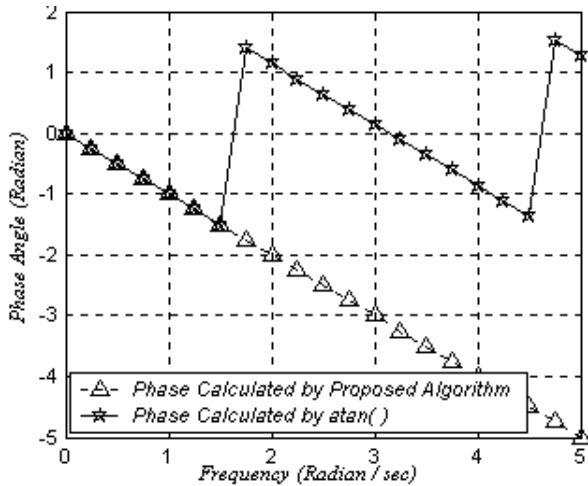**Figure 3.** Flowchart of the proposed algorithm

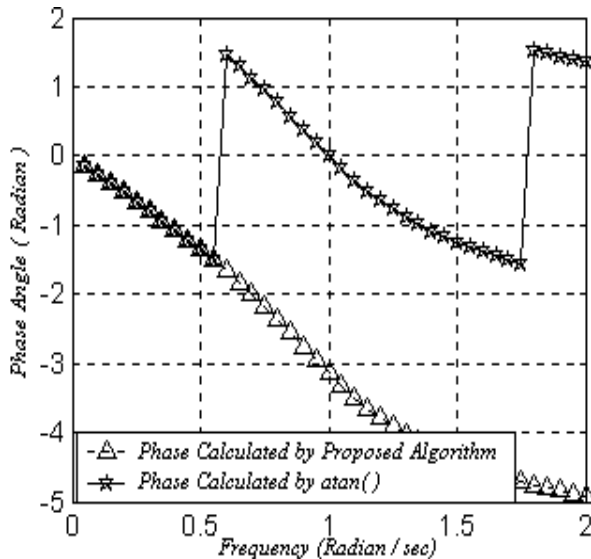**Figure 4.** Phase characteristic of All-Pass filter

Figure 5. Phase characteristic of analog 4-though order LPF

To compare the computational benefits of the proposed algorithm with that of MAT-LAB function UNWRAP we repeated phase characteristic calculations of the upper 25-Th order low-pass FIR filter several times varying the number of points in the complex array. And each time we calculated the relative CPU time units used by proposed algorithm and by UNWRAP function. The relative CPU time units were calculated using MATLAB function CPUTIME. In Figure 8 is shown the dependence of the relative CPU time units used

by both algorithms on the number of points to be calculated. From Figure 8 we can see that the time needed for UNWRAP function increases very fast by increasing the number of points, the relation very similar to quadratic relation. While the relation for the proposed algorithm is similar to linear relation. Which means that proposed algorithm is more efficient than UNWRAP function especially for large arrays of complex numbers.

Figure 9 depicts the dependence of relative CPU time units saved by using proposed algorithm on the size of complex array for which the phase to be calculated. It shows that CPU time saved increases by quadratic law as the size of complex array increases.
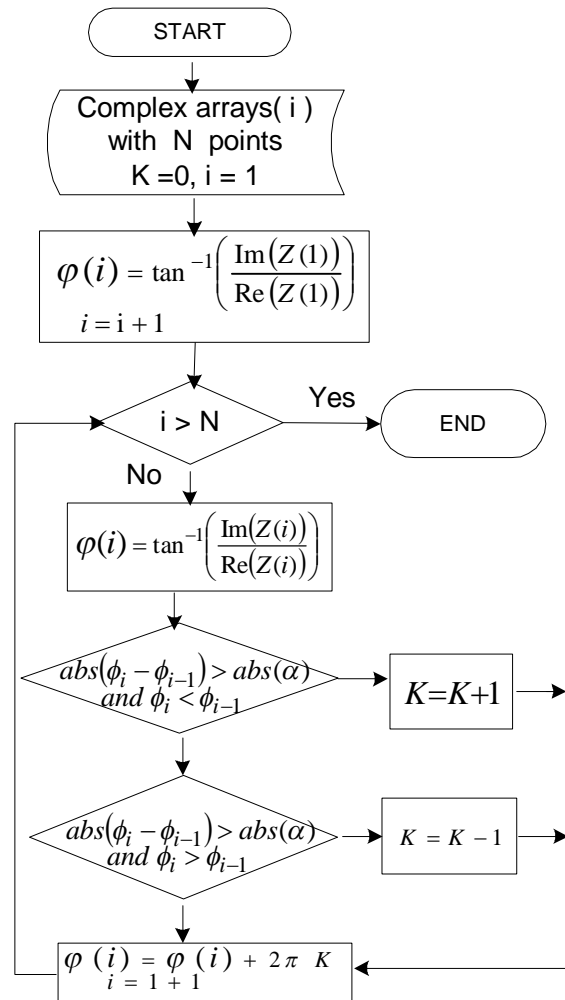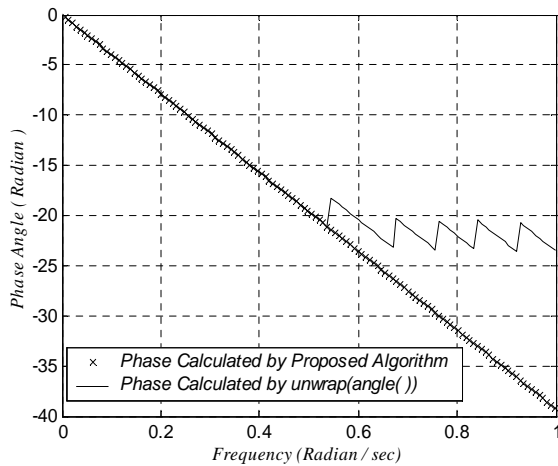

Figure 6. Unwrap algorithm flowchart

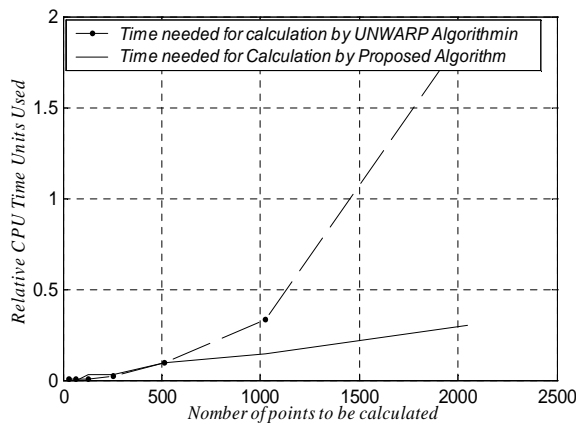**Figure 7.** Phase characteristic of 25-though order LPF FIR filter

Figure 8. Algorithms speed comparison

tion UNWRAP which tries to detect branch cut crossings after implementing arctangent function which leads to eliminating natural jumps in some cases which is not acceptable.

- The proposed algorithm always distinguishes between natural jumps and an artifact of the arctangent function inside angle.

- The proposed algorithm can not be fooled by sparse, rapidly changing phase values as MATLAB functions UNWRAP.

- The proposed algorithm is more computational efficient than MATLAB functions UNWRAP especially for large complex arrays.
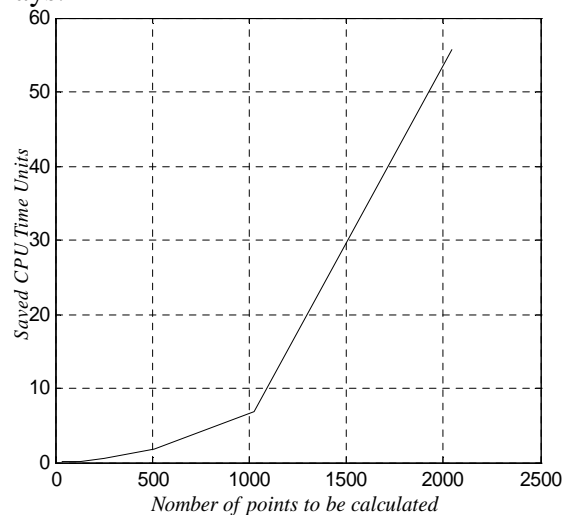

Figure 9. CPU time saved by using proposed algorithm
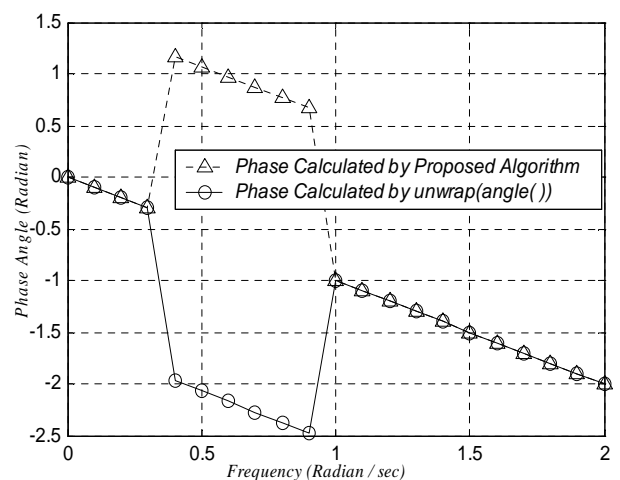
Figure 10. Phase charecterristic with natural jumps

To check the behavior of proposed algorithm on natural jumps, we inserted a natural jump of $\pi/2$ in part of all-pass filter complex transfer function array (range of frequency $0.4 < \omega \le 0.9$ as shown in Figure 10) and then built the phase characteristic of the array using both proposed algorithm and MATLAB function UNWRAP. From Figure 10 it can be seen that proposed algorithm built phase characteristic with correct phase jumps as inserted in the array.

Comparing proposed algorithm with UNWRAP algorithm, the following advantages of proposed algorithm may be stated;

- It builds the phase characteristic directly from the data and not like MATLAB func-

6. Conclusions

A new accurate and efficient method for phase unwrapping is proposed. The method is based on recurrent calculation and summation of phase difference between neighboring points. This method is easy, requires minimum computation and no basic limitations on its application. The tests performed demonstrate the validity of this approach. Also the algorithm was tested with functions that have real jumps in phase and the resulting phase characteristics of these functions were calculated and plotted correctly.

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