# Vibration Characteristics of Rectangular Plates Resting on Elastic Foundations and Carrying any Number of Sprung Masses 

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#### Abstract

A new version differential quadrature method is proposed to obtain the vibration characteristics of rectangular plates resting on elastic foundations and carrying any number of sprung masses. The accuracy of the technique is demonstrated by comparing the calculated results with the published data. The non-uniform grid spacing is used in this work. The results also demonstrate the efficiency of the method in treating the vibration problem of the rectangular plates carrying any number of sprung masses and resting on the elastic foundations.


Keywords: the differential quadrature method; rectangular plates; numerical methods; sprung masses; elastic foundations; vibration analysis.

## 1. Introduction

Vibration is the most important modes of failure in plates it plays a crucial role in engineering. Leissa [1,2] derived exact solutions for the free vibration and buckling problems of the rectangular plates. Boay [3] analyzed the natural frequencies of plates with and without a concentrated mass using the Rayleigh-energy method. Avalos et al. [4] dealt with the solution of vibration by a simply mounted concentred mass using the well-known normal mode. Xiang et al. [5] used the Ritz method combined with a variation to solve vibration of rectangular mindlin plates resting on elastic edge supports. Laura and Grossi $[6,7]$ calculated the fundamental frequency coefficient for a rectangular plate with edges elastically restrained against both translation and rotation using polynomial coordinate functions and the Rayleigh-Ritz's
method. Wu and Luo [8] solved the problem of the natural frequencies and the corresponding mode shapes of a uniform rectangular flat plate carrying any number of point masses and translational springs using the analytical -and-numerical-combined method. Nicholson and Bergman [9] used the Green's function express the natural modes for the damped plate-oscillator systems. Gorman [10] solved the free vibration problem of shear deformable plates resting on uniform elastic foundations using the modified Superpo-sion-Galerkin method. This work focuses on the application of differential quadrature method to the vibration of plates resting on the elastic foundations and carrying any number of sprung masses. In the following section an overview of differential quadrature method to preset the computation of its weighting coefficients offered and discussed the selection problem. The integrity and computational ef-
ficiency of the method will be demonstrated through a series of case studies. Very few papers in the literature have presented the vibration analysis of rectangular plates resting on elastic foundations and carrying any number of sprung masses using the differential quadrature method.

## 2. Basic equation

The strain energy of the rectangular plate resting on the elastic foundation and carrying any number of sprung masses is given by
$U=\frac{1}{2} \int_{0}^{b} \int_{0}^{a} D\left(\frac{\partial^{2} w(x, y, t)}{\partial x^{2}}+\frac{\partial^{2} w(x, y, t)}{\partial y^{2}}\right)^{2} d x d y$
$+\frac{1}{2} \int_{0}^{b} \int_{0}^{a} k_{g}(w(x, y, t))^{2} d x d y$
$+\sum_{i=1}^{s} \frac{1}{2} k_{i}\left(z_{i}(t)-w\left(x_{i}, y_{i}, t\right)\right)^{2}$
where $k_{g}$ is the foundation stiffness . The kinetic energy of the rectangular plates resting on the elastic foundation and carrying any number of sprung masses is given by

$$
\begin{equation*}
T=\frac{1}{2} \rho h \int_{0}^{b} \int_{0}^{a}\left(\frac{\partial w(x, y, t)}{\partial t}\right)^{2} d x d y+\sum_{i=1}^{s} \frac{1}{2} M_{i}\left(\frac{\partial z_{i}(t)}{\partial t}\right)^{2} \tag{2}
\end{equation*}
$$

where $s$ is the total number of sprung masses attached to the rectangular plate, $w$ is the deflection of the rectangular plate, $M_{i}$ is the mass of the $i^{\text {th }}$ oscillator, $k_{i}$ is the spring constant of the $i^{\text {th }}$ oscillator, $z_{i}$ is the sprung mass location of the $i^{\text {th }}$ oscillator, $x_{i}$ is the location of sprung mass of the $i^{\text {th }}$ oscillator in the direction of $x$-axis, $y_{i}$ is the location of sprung mass of the $i^{\text {th }}$ oscillator in the direction of y-axis, $t$ is the time, $D=E h^{3} /\left(12\left(1-v^{2}\right)\right)$ is the flexural rigidity, E is Young's modulus, $\rho$ is the density of the plate material, and h is the rectangular plate thickness. With considering the internal and external damping effects in the rectangular plate, the virtual work $\delta W$ in
the plate can be derived as

$$
\begin{align*}
& \delta W=-\int_{0}^{b} \int_{0}^{b} C_{0} \frac{\partial w(x, y, t)}{\partial t} \delta w d x d y-\int_{0}^{b} \int_{D}^{a} C_{D} D \frac{\partial^{5} w(x, y, t)}{\partial x^{4} \partial t} \delta w d x d y \\
& -2 \int_{\partial}^{b} \int_{0}^{a} C_{D} D \frac{\partial^{5} w(x, y, t)}{\partial x^{2} \partial y^{2} \partial t} \delta w d x d y-\int_{0}^{a} \int_{0} C_{l} D \frac{\partial^{5} w(x, y, t)}{\partial y^{4} \partial t} \delta w d x d y \tag{3}
\end{align*}
$$

where $C_{0}$ and $C_{1}$ are the external and the internal damping coefficients of the rectangular plate, respectively. Substituting Equations (1), (2), and (3) into Hamilton equation

$$
\begin{equation*}
\int_{t_{1}}^{t_{2}}(\delta T-\delta U+\delta W) d t=0 \tag{4}
\end{equation*}
$$

This leads to the equations of motion of the rectangular plate on the elastic foundation with any number of sprung masses as

$$
\begin{align*}
& D \frac{\partial^{4} w(x, y, t)}{\partial x^{4}}+2 D \frac{\partial^{4} w(x, y, t)}{\partial x^{2} \partial y^{2}}+D \frac{\partial^{4} w(x, y, t)}{\partial y^{4}} \\
& +k_{g} w(x, y, t)-{ }_{i=0}^{s} k_{i}\left(z_{i}(t)-\right. \\
& \left.w\left(x_{i}, y_{i}, t\right)\right) \delta\left(x-x_{i}\right) \delta\left(y-y_{i}\right) \\
& +C_{0} \frac{\partial w(x, y, t)}{\partial t} \\
& +C_{1} D \frac{\partial^{5} w(x, y, t)}{\partial x^{4} \partial t}+2 C_{1} D \frac{\partial^{5} w(x, y, t)}{\partial x^{2} \partial y^{2} \partial t} \\
& +C_{1} D \frac{\partial^{5} w(x, y, t)}{\partial y^{4} \partial t} \\
& +\rho h \frac{\partial^{2} w(x, y, t)}{\partial t^{2}}=0  \tag{5}\\
& M_{i} \frac{d^{2} z_{i}(t)}{d t^{2}}+k_{i}\left(z_{i}(t)-w\left(x_{i}, y_{i}, t\right)\right)=0 \tag{6}
\end{align*}
$$

for $i=1,2, \ldots, s$
At a simply supported boundary, the transverse deflection of the rectangular plate can be expressed as

$$
\begin{align*}
& w(0, y, t)=0 \\
& w(x, b, t)=0 \tag{7}
\end{align*}, \quad w(a, y, t)=0, \quad w(x, 0, t)=0
$$

The condition of zero normal moment can be reduced to
$\frac{\partial^{2} w(0, y, t)}{\partial x^{2}}=0$
$\frac{\partial^{2} w(a, y, t)}{\partial x^{2}}=0 \quad \frac{\partial^{2} w(x, 0, t)}{\partial y^{2}}=0$,
$\frac{\partial^{2} w(x, b, t)}{\partial y^{2}}=0$
Substituting $w(x, y, t)=W(x, y) e^{\lambda t} \quad$ and $z_{i}=Z_{i} e^{\lambda t}$ into Equations (5) and (6), Equations (5) and (6) can be written as

$$
\begin{align*}
& D \frac{\partial^{4} W(x, y)}{\partial x^{4}}+2 D \frac{\partial^{4} W(x, y)}{\partial x^{2} \partial y^{2}}+D \frac{\partial^{4} W(x, y)}{\partial y^{4}} \\
& +k_{g} W(x, y)-\sum_{i=1}^{s} k_{i}\left(Z_{i}-W\left(x_{i} y_{i}\right)\right) \delta\left(x-x_{i}\right) \delta\left(y-y_{i}\right) \\
& +\lambda C_{0} W(x, y) \\
& +\lambda C_{1} D \frac{\partial^{4} W(x, y)}{\partial x^{4}}+2 \lambda C_{1} D \frac{\partial^{4} W(x, y)}{\partial x^{2} \partial y^{2}}+\lambda C_{1} D \frac{\partial^{4} W(x, y)}{\partial y^{4}} \\
& +\lambda^{2} \rho h W(x, y)=0 \tag{9}
\end{align*}
$$

$k_{i}\left(Z_{i}-W\left(x_{i}, y_{i}\right)\right)+\lambda^{2} M_{i} Z_{i}=0$
for $i=1,2, \ldots, s$

## 3. Method of solution

With the increasing use of new fast and affordable computers, along with the availability of various numerical methods, the solutions of several complicated engineering problems have now become efficiently achievable. The finite differences method, Rayleigh-Ritz method, the finite element method, Fourier series method and the boundary element method have been used extensively for solving linear and nonlinear differential equations, and consequently there are
several commercially developed software packages. The development of new techniques from the standpoint of computational efficiency and numerical accuracy is of primal interest. Since it has been developed, several researchers have applied successfully the differential quadrature method to solve a variety of problems in different fields of science and engineering [11-17]. The partial differential equation can be reduced to a set of algebraic equations using the differential quadrature method. The differential quadrature method uses the basis of the Gauss method in deriving the derivative of a function. It follows that the partial derivative of a function with respect to a space variable can be approximated by a weighted linear combination of function values at some intermediate points in that variety. A differential quadrature approximation at the ith discrete point on a grid in the direction of x -axis is given by

$$
\begin{align*}
& \frac{\partial^{m} f\left(x_{i}, y\right)}{\partial x^{m}}=\sum_{j=1}^{N_{x}} A_{i j}^{(m)} f\left(x_{j}, y\right) \\
& \text { for } i=1,2, \ldots, N_{x} \tag{11}
\end{align*}
$$

A differential quadrature approximation at the $i^{\text {th }}$ discrete point on a grid in the direction of $y$-axis may be written as

$$
\begin{equation*}
\frac{\partial^{m} f\left(x, y_{i}\right)}{\partial y^{m}}=\sum_{j=1}^{N_{v}} B_{i j}^{(m)} f\left(x, y_{j}\right) \tag{12}
\end{equation*}
$$

for $i=1,2, \ldots, N_{y}$
where $A_{i j}^{(m)}$ and $B_{i j}^{(m)}$ are the weighting coefficients. The test function can be written as
$f(x, y)=x^{\alpha-1} y^{\beta-1}$
for $\alpha=1,2, \ldots, N_{x}$ and $\beta=1,2, \ldots, N_{y}$
Substituting Equation (13) to Equations (11) and (12), Equations (11) and (12) are computed by

$$
\sum_{k=1}^{N_{x}} x_{k}^{\alpha-1} A_{i k}^{(m)}=\left.\frac{\partial^{m} x^{\alpha-1}}{\partial x^{m}}\right|_{x=x_{i}}
$$

$$
\begin{equation*}
\text { for } i=1,2, \ldots, N_{x} \text { and } \alpha=1,2, \ldots, N_{x} \tag{14}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{k=1}^{N_{y}} y_{l}^{\beta-1} B_{j k}^{(m)}=\left.\frac{\partial^{m} y^{\beta-1}}{\partial y^{m}}\right|_{y=y_{j}} \\
& \text { for } j=1,2, \ldots, N_{y} \text { and } \beta=1,2, \ldots, N_{y} \tag{15}
\end{align*}
$$

The $m^{t h}$ order derivates may be obtained using the following equations

$$
\begin{equation*}
A_{i j}^{(m)}=\sum_{k=1}^{N_{x}} A_{i k}^{(1)} A_{k j}^{(m-1)} \tag{16}
\end{equation*}
$$

where $A_{i j}^{(m)}$ is the higher order weighting coefficient matrix in the direction of $x$-axis.

$$
\begin{equation*}
B_{i j}^{(m)}=\sum_{k=1}^{N_{y}} B_{i k}^{(1)} B_{k j}^{(m-1)} \tag{17}
\end{equation*}
$$

where $B_{i j}^{(m)}$ is the $m^{\text {th }}$ order weighting coefficient matrix in the direction of $y$-axis. The above relation gives the higher order weighting coefficient matrix based on the first-order derivative weighting coefficients. The above relations are not restricted to the choice of sampling points. It is emphasized that the number of the test functions must be greater than the highest order of derivative in the governing equations. The selection of locations of the sampling points is important in ensuring the accuracy of the solution of differential equations. A more accurate solution could be obtained by choosing a set of unequally spaced sampling points for a domain separate into by $N_{x}$ and $N_{y}$ points. A simple and good choice can be the roots of shifted Chebyshev and Legendre points. The inner points are
$x_{i}=\frac{a}{2}\left(1-\cos \frac{(i-1) \pi}{N_{x}-1}\right)$
for $i=1,2, \ldots, N_{x}$
in the direction of x -axis
$y_{i}=\frac{b}{2}\left(1-\cos \frac{(i-1) \pi}{N_{y}-1}\right)$
for $i=1,2, \ldots, N_{y}$
in the direction of $y$-axis. $a$ is the length of the plate in the direction of $x$-axis and $b$ is the length of the plate in the direction of y-axis. Substituting Equations (14) and (15) to Equations (9) and (10), leads to

$$
\begin{aligned}
& D \sum_{k=1}^{N_{x}} A_{i k}^{(4)} W_{k j}+2 D \sum_{k=1}^{N_{x}} A_{i k}^{(2)} \sum_{l=1}^{N_{v}} B_{j l}^{(2)} W_{k l}+D \sum_{l=1}^{N_{v}} B_{j l}^{(4)} W_{i l} \\
& +k_{g} W_{i j}-\sum_{i=1}^{s} k_{i}\left(Z_{i}-W\left(x_{i} y_{i}\right)\right) \delta\left(x-x_{i}\right) \delta\left(y-y_{i}\right) \\
& +\lambda C_{0} W_{i j}
\end{aligned}
$$

$$
+\lambda C_{1} D \sum_{k=1}^{N_{c}} A_{i k}^{(4)} W_{k j}+2 \lambda C_{1} D \sum_{k=1}^{N_{k}} A_{k j}^{(2)} \sum_{l=1}^{N_{v}} B_{j l}^{(2)} W_{k l}+\lambda C_{1} D \sum_{l=1}^{N_{n}} B_{i l}^{(4)} W_{i l}
$$

$$
+\lambda^{2} \rho h W_{i j}=0
$$

$$
\begin{equation*}
\text { for } i=1,2, \ldots, N_{x} \text { and } j=1,2, \ldots, N_{y} \tag{20}
\end{equation*}
$$

$k_{i}\left(Z_{i}-W\left(x_{i}, y_{i}\right)\right)+\lambda^{2} M_{i} Z_{i}=0$

$$
\begin{equation*}
\text { for } i=1,2, \ldots, s \tag{21}
\end{equation*}
$$

The transverse deflection of the plate at a simply supported boundary can be written as
$W_{1 j}=0$ for $j=1,2, \ldots, N_{y}$,
$W_{N_{x} j}=0$ for $j=1,2, \ldots, N_{y}$,
$W_{i 1}=0$ for $i=1,2, \ldots, N_{x}$,
$W_{i N_{y}}=0$ for $i=1,2, \ldots, N_{x}$
The condition of zero normal moment can be reduced to the following discrete forms.
$\sum_{l=1}^{N_{x}} A_{l l}^{(2)} W_{l j}=0$ for $j=1,2, \ldots, N_{y}$,
$\sum_{l=1}^{N_{x}} A_{N_{x}}^{(2)} W_{l j}=0$ for $j=1,2, \ldots, N_{y}$,
$\sum_{l=1}^{N_{v}} B_{l l}^{(2)} W_{l j}=0$ for $j=1,2, \ldots, N_{x}$,
$\sum_{l=1}^{N_{v}} B_{N_{l}}^{(2)} W_{l j}=0$ for $j=1,2, \ldots, N_{x}$

## 4. Numerical results

Figure 1 shows the eigenvalue of the plates that are supported, as all of edges are simply supported with various values of $\mathrm{a} / \mathrm{b}$. The data used in this analysis are as follows:
$h=0.005 \mathrm{~m}, v=0.3, \rho h=39.25 \mathrm{~kg} / \mathrm{m}^{2}$,
$E=2.051 \times 10^{11} \mathrm{~N} / \mathrm{m}^{2}$,
$D=E h^{3} /\left[12\left(1-v^{2}\right)\right]=2.3478 \times 10^{3} N \times m$,
$C_{1}=0.0, C_{0}=0.0, M_{P}=\rho h a b=2355 \mathrm{~kg}$,
$M_{i} / M_{p}=0.0$,
$k_{P}=D / a^{2}=5.8695 \times 10^{2} \mathrm{~N} / \mathrm{m}$,
$k_{i} / k_{p}=0.0$, and $k_{g} / k_{p}=0.0$.
The dimensionless natural frequency is defined as $\Omega=\omega a^{2} \sqrt{(\rho h / D)}$ The numerical results in figure 1 are cited from reference [1]. The results reveal that natural frequencies $\Omega$ increase as the values of $\mathrm{a} / \mathrm{b}$ increase. It can be seen that the numerical results agree well with the data from theory [1]. The comparisons and numerical examples show the effectiveness of
the differential quadrature method. The differential quadrature method can serve as a useful tool to obtain dynamic behavior of the rectangular plates. Figure 2 plots the eigenvalue of the plates that are supported, as all of edges are simply supported with the roots of shifted Chebyshev and Legendre points. The geometrical and material data used in this analysis are as follows:

$$
\begin{aligned}
& a=1.0 \mathrm{~m}, \quad b=1.0 \mathrm{~m}, \quad x_{1}=0.5 \mathrm{~m}, y_{1}=0.5 \mathrm{~m}, \\
& h=0.005 \mathrm{~m}, \quad v=0.3, \quad \rho h=39.25 \mathrm{~kg} / \mathrm{m}^{2}, \\
& E=2.051 \times 10^{11} \mathrm{~N} / \mathrm{m}^{2}, \\
& D=E h^{3} /\left[12\left(1-v^{2}\right)\right]=2.3478 \times 10^{3} \mathrm{~N} \times \mathrm{m}, \\
& C_{1}=0.0, C_{0}=0.0, M_{P}=\rho h a b=2355 \mathrm{~kg}, \\
& M_{1} / M_{p}=0.001, \\
& k_{P}=D / a^{2}=5.8695 \times 10^{2} \mathrm{~N} / \mathrm{m}, \\
& \text { and } k_{1} / k_{p}=100.0 .
\end{aligned}
$$

Analysis of the numerical results reveals that natural frequencies $\Omega$ increase as foundation stiffnesses increase. The results show that foundation stiffnesses have larger influences on the first, second and third natural frequencies than the other natural frequencies.

## 5. Conclusions

The differential quadrature method is shown an efficient way of obtaining accurate solutions to the problem of rectangular plates resting on elastic foundations and carrying any number of sprung masses. The contrast and numerical results solved using shift Chebyshev and Legendre points show the effectiveness of the differential quadrature method. The excellent agreements are found between the proposed scheme and known solutions published in the literature.


Figure 1. The first six natural frequencies of the plates for various values of $\mathrm{a} / \mathrm{b}$.


Figure 2. The first six natural frequencies of the plates for various foundation stiffnesses.

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