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A Note on a Three-Dimensional Bäcklund Transformation

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Abstract: The Bäcklund transformation (BT) for a three-dimensional nonlinear wave equation and its nonlinear superposition formula are studied in this note. We prove that the three dimensional Bäcklund transformation obtained by Leibbrandt, *et al*. can be decomposed into three two-dimensional BTs. Some results on the N-dimensional Liouville equation are also discussed in the article.

Keywords: Bäcklund transformation; Liouville equation; superposition formula

1. Introduction

For the past decades the study of nonlinear wave equations has attracted a lot of attentions from scientists and mathematicians. Some powerful tools such as the singular perturbation, the inverse scattering transform, the Bäcklund transformation (BT), *etc*., are developed in solving these equations. It is interesting to note that the Bäcklund transformation was first introduced in pseudo-spherical surface but now is very useful in nonlinear equations.

Liouville equation in three dimensions takes the form:

$$\nabla^2\alpha = \exp\alpha, \quad \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2, \tag{1}$$

with the following conditions:

$$\alpha \to -\infty, \ \frac{d\alpha}{dr} \to 0, \ r \equiv (x^2+y^2+z^2)^{\frac{1}{2}} \to +\infty \tag{2}$$

In two-dimensional space, this equation is reduced to

$$(\partial_x^2 + \partial_y^2)\chi = k\exp(a\chi) \tag{3}$$

with some boundary condition on χ, where χ is a scalar field, and a, k are real constants.

Equation (3) was first obtained by Liouville, and was studied later by many well-known mathematicians that include Picard, Poincare and Bierberbach, *etc*. The equation has significant applications in electro-statistics, hydrodynamics, cosmology, isothermal gas spheres and monopole theory.

A Bäcklund transformation for equation (1) and its nonlinear superposition formula was proposed by Leibbrandt, *et al*. [3,4]. In this note, we prove that the three-dimensional Bäcklund transformation of [3,4] can be decomposed into three two-dimensional BTs. We also discuss some results for the N-dimensional Liouville equation. Since the theory of Bäcklund transformations is still very active [6,8], thus this discussion has, ob-

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viously, some interests.

2. Decomposition of Bäcklund transformation for the Liouville equation

The Bäcklund transformation derived by Leibbrant et al. for the Liouville equation in three spatial dimensions is

$$K(i\beta-\alpha)=\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right)\exp i\theta\wp \quad (4)$$

$$K\equiv I\partial_x+i(\sigma_1\partial_y+\sigma_3\partial_z) \quad (5)$$

$$\wp\equiv\sigma_1\exp(-i\lambda\sigma_2) \quad (6)$$

where,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are Pauli matrices and I the 2×2 identity matrix, θ,λ $(0\le\theta\le2\pi,\quad 0\le\lambda\le2\pi)$ are the parameters of the BT, α and β are real functions satisfying Equation (1) and the Laplace Equation:

$$\nabla^2\beta=0,\quad \nabla^2=\partial_x^2+\partial_y^2+\partial_z^2 \quad (7)$$

respectively.

Note that

$$\exp(i\lambda\sigma_2)=I\cos\lambda+i\sigma_1\sigma_2\sin\lambda \quad (8)$$

$$\exp\left\{\theta\begin{pmatrix}i\sin\lambda & i\cos\lambda\\ i\cos\lambda & -\sin\lambda\end{pmatrix}\right\}=I\cos\theta+\sin\theta\begin{pmatrix}i\sin\lambda & i\cos\lambda\\ i\cos\lambda & -i\sin\lambda\end{pmatrix} \quad (9)$$

We can prove that the Bäcklund transformation of (1) takes the following form:

$$K(i\beta-\alpha)=(A+iB)\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (10)$$

where

$$A=I\cos\theta \quad (11)$$

$$B=\begin{pmatrix}\sin\theta\sin\lambda & \sin\theta\cos\lambda\\ \sin\theta\cos\lambda & -\sin\theta\sin\lambda\end{pmatrix} \quad (12)$$

This matrix equation implies the four Equations:

$$(\partial_x+i\partial_z)(i\beta-\alpha)=(cos\theta+i\sin\theta\sin\lambda)\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (13a)$$

$$(\partial_x-i\partial_z)(i\beta-\alpha)=(cos\theta-i\sin\theta\sin\lambda)\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (13b)$$

$$i\partial_y(i\beta-\alpha)=i\sin\theta\cos\lambda\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (13c)$$

$$i\partial_y(i\beta-\alpha)=i\sin\theta\cos\lambda\sqrt{2}\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (13d)$$

or, their equivalent forms:

$$\partial_x(i\beta-\alpha)=\sqrt{2}\cos\theta\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (14a)$$

$$\partial_y(i\beta-\alpha)=\sqrt{2}\sin\theta\cos\lambda\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (14b)$$

$$\partial_z(i\beta-\alpha)=\sqrt{2}\sin\theta\sin\lambda\exp\left(\frac{\alpha+i\beta}{2}\right) \quad (14c)$$

If we compare the real and imaginary parts of the equations in (14), and use the integral conditions, we can show that equations (14a,b) are Bäcklund transformation for the two- dimensional Liouville equation in the light-cone coordinate system:

$$\alpha_{xy}=\sin\theta\cos\theta\cos\lambda\exp\alpha \quad (15)$$

and the wave equation $\beta_{xy}=0$. Similarly, equations (14a,c) are BT for

$$\alpha_{xz}=\sin\theta\cos\theta\sin\lambda\exp\alpha \quad (16)$$

and the wave equation $\beta_{xz}=0$; and equations (14b,c) are BT for

$$\alpha_{yz}=\sin^2\theta\sin\lambda\cos\lambda\exp\alpha \quad (17)$$

and $\beta_{yz} = 0$

We would like to point out that:

1). The equations (15), (16), and (17) can be written, with some coordinate transformations, as the standard form as in (3).

2). From (14) we have

$$\alpha_{xx} = \cos^2\theta \exp\alpha \qquad (18a)$$

$$\alpha_{yy} = \sin^2\theta \cos^2\lambda \exp\alpha \qquad (18b)$$

$$\alpha_{zz} = \sin^2\theta \sin^2\lambda \exp\alpha \qquad (18c)$$

and thus

$$\alpha_{xx} + \alpha_{yy} + \alpha_{zz} = \exp\alpha \qquad (19)$$

which is the original three-dimensional Liouville equation. Similarly, we have

$$\beta_{xx} + \beta_{yy} + \beta_{zz} = 0 \qquad (20)$$

which is Laplace equation (7). Therefore, we can see that the three sets of BTs for two-dimensional Liouville equations also satisfy the conditions for the three-dimensional Liouville equation (1).

In Bäcklund transformation (14), since the parameters $\theta, \lambda \quad (0 \le \theta \le 2\pi, \quad 0 \le \lambda \le 2\pi)$ are real, all the trigonometric functions in (14) are real. Therefore, we have shown that the Bäcklund transformation (14) for the three-dimensional Liouville equation (1) is able to be decomposed into three sets of BTs for two-dimensional Liouville equations. Similar results can be obtained for the Liouville equation in N spatial dimensions.

3. Some results of Liouville equations in N spatial dimensions

We write (14) into the following:

$$\partial_{x^{(1)}}(i\beta - \alpha) = a_1\sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right) \qquad (21a)$$

$$\partial_{x^{(2)}}(i\beta - \alpha) = a_2\sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right) \qquad (21b)$$

$$\partial_{x^{(3)}}(i\beta - \alpha) = a_3\sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right) \qquad (21c)$$

where $a_j = a_j(\theta, \lambda) \quad (j = 1, 2, 3)$ are the parameters of the transformation, which satisfy

$$a_1^2 + a_2^2 + a_3^2 = 1 \qquad (22)$$

It is easy to see that $\alpha_j, j = 1, 2, 3$ are real due to the ranges of *θ, λ*.

Consider the transformation in N-dimensional space

$$\partial_{x^{(j)}}(i\beta - \alpha) = a_j\sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right) \quad (j = 1, 2, \ldots N)$$

where $\alpha_j, j = 1, 2, 3, \ldots, N,$ satisfy $\sum_{j=1}^{N} a_j^2 = 1$.

Obviously, α and β in (23) satisfy the N-dimensional Liouville equation

$$\sum_{j=1}^{N} \partial^2_{x^{(j)}} \alpha = \exp\alpha \qquad (24)$$

and the N-dimensional Laplace equation

$$\sum_{j=1}^{N} \partial^2_{x^{(j)}} \beta = 0 \qquad (25)$$

Let $\varepsilon_1 = (a_1, a_2 \ldots a_N)$, and extend it to a standard orthogonal basis of the N-dimensional space R^N: $\{\varepsilon_1, \varepsilon_2 \ldots \varepsilon_N\}$. Consider the matrix A with the low vectors of $\varepsilon_j (j = 1.2 \ldots N)$, and the coordinate transformation:

$$\left[y^{(1)} y^{(2)} \ldots y^{(N)}\right]^T = A\left[x^{(1)} x^{(2)} \ldots x^{(N)}\right]^T \qquad (26)$$

We have

$$\partial_{y^{(1)}}(i\beta - \alpha) = \sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right), \qquad (27)$$

$$\partial_{y^{(j)}}(i\beta - \alpha) = 0, \quad (j = 2, 3, \ldots, N) \qquad (28)$$

This means that under the new coordinates, α and β are only related to $y^{(1)}$, which satisfy the regular easily solvable two-dimensional

Liouville and wave equations:

$$\alpha_{y^{(1)}y^{(1)}} = \exp\alpha, \tag{29}$$

$$\beta_{y^{(1)}y^{(1)}} = 0. \tag{30}$$

For the Liouville equation with time variable t, we have a similar result. For example, consider

$$(\nabla^2 - \partial_t^2)\alpha = \exp\alpha, \text{ where } \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2, \tag{31}$$

and its Bäcklund transformation

$$\overline{K}(i\beta - \alpha) = \sqrt{2}\exp\left(\frac{\alpha + i\beta}{2}\right)\exp i\theta\overline{\wp} \tag{32}$$

where

$$\overline{K} \equiv I\partial_x + i(\sigma_1\partial_y + \sigma_3\partial_z) + \sigma_2\partial_t, \tag{33}$$

$$\overline{\wp} \equiv \sigma_1\exp[(-i\lambda\sigma_2)\exp(-\tau\sigma_1)],\ (-\infty < \tau < +\infty) \tag{34}$$

and β satisfies the equations

$$(\nabla^2 - \partial_t^2)\beta = 0, \text{ where } \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{35}$$

It can be proved that the Bäcklund transformation (32) with time variable t has a similar structure with the one of (4) without time t.

4. Nonlinear superposition formula of solutions

In references [3,4], the following nonlinear superposition formula of solutions for Liouville equation of higher dimensions have been used:

$$\tan\left(\frac{\beta_2 - \beta_0}{4}\right) = R_{12}\tanh\left(\frac{\alpha_1^{(1)} - \alpha_1^{(2)}}{4}\right), \tag{36}$$

$$R_{12} = \pm\left[(1+\aleph_{12})/(1-\aleph_{12})\right]^{\frac{1}{2}}, \quad |\aleph_{12} < 1|, \tag{37}$$
$$\aleph_{12} \equiv \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\lambda_1 - \lambda_2),$$

where $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, β_0 and β_2 are the real solutions of equations (1) and (7), respectively. These solutions are connecting by the Bäcklund transformation (4). Assume that the BT (4) is permutable, we have

$$i\beta_0 \xrightarrow{\theta_1,\lambda_1} \alpha_1^{(1)} \xrightarrow{\theta_2,\lambda_2} i\beta_2, \quad i\beta_0 \xrightarrow{\theta_2,\lambda_2} \alpha_1^{(2)} \xrightarrow{\theta_1,\lambda_1} i\beta_2 \tag{38}$$

We will show that the above superposition formula is trivial. According to (21), we can change (38) into

$$i\beta_0 \xrightarrow{a_j} \alpha_1^{(1)} \xrightarrow{a_j^{'}} i\beta_2, \quad i\beta_0 \xrightarrow{a_j^{'}} \alpha_1^{(2)} \xrightarrow{a_j} i\beta_2 \tag{39}$$

where α_j, $\alpha_j^{'}$ (j = 1,2,3) depend on θ_j, λ_j, It follows that α_j, $\alpha_j^{'}$ are real due to the range of θ_j and λ_j. We have

$$\partial x^{(j)}(i\beta_0 - \alpha_1^{(1)}) = a_j\sqrt{2}\exp\left(\frac{\alpha_1^{(1)} + i\beta_0}{2}\right) \tag{40a}$$

$$\partial x^{(j)}(i\beta_0 - \alpha_1^{(2)}) = a_j^{'}\sqrt{2}\exp\left(\frac{\alpha_1^{(2)} + i\beta_0}{2}\right) \tag{40b}$$

$$\partial x^{(j)}(\alpha_1^{(1)} - i\beta_2) = a_j^{'}\sqrt{2}\exp\left(\frac{i\beta_2 + \alpha_1^{(1)}}{2}\right) \tag{40c}$$

$$\partial x^{(j)}(\alpha_1^{(2)} - i\beta_2) = a_j\sqrt{2}\exp\left(\frac{i\beta_2 + \alpha_1^{(2)}}{2}\right), \quad (j = 1,2,3) \tag{40d}$$

From (40a,c) and (40b,d), we obtain

$$\partial x^{(j)}(i\beta_0 - i\beta_2) = a_j\sqrt{2}\exp\left(\frac{\alpha_1^{(1)} + i\beta_0}{2}\right) + a_j^{'}\sqrt{2}\exp\left(\frac{i\beta_2 + \alpha_1^{(1)}}{2}\right) \tag{41}$$

and

$$\partial x^{(j)}(i\beta_0 - i\beta_2) = a_j^{'}\sqrt{2}\exp\left(\frac{\alpha_2^{(2)} + i\beta_0}{2}\right) + a_j\sqrt{2}\exp\left(\frac{i\beta_2 + \alpha_1^{(2)}}{2}\right) \tag{42}$$

Also,

$$a_j\sinh\left(\frac{\alpha_1^{(1)} - \alpha_1^{(2)} + i\beta_0 - i\beta_2}{4}\right)\left[\cosh\left(\frac{\alpha_1^{(1)} + \alpha_1^{(2)} + i\beta_0 + i\beta_2}{4}\right) + \sinh\left(\frac{\alpha_1^{(1)} + \alpha_1^{(2)} + i\beta_0 + i\beta_2}{4}\right)\right]$$
$$= a_j^{'}\sinh\left(\frac{\alpha_1^{(2)} - \alpha_1^{(1)} + i\beta_0 - i\beta_2}{4}\right)\left[\cosh\left(\frac{\alpha_1^{(1)} + \alpha_1^{(2)} + i\beta_0 + i\beta_2}{4}\right) + \sinh\left(\frac{\alpha_1^{(1)} + \alpha_1^{(2)} + i\beta_0 + i\beta_2}{4}\right)\right] \tag{43}$$

or equivalently,

$$a_j \sinh\left(\frac{\alpha_1^{(1)}-\alpha_1^{(2)}+i\beta_0-i\beta_2}{4}\right)\exp\left(\frac{\alpha_1^{(1)}+\alpha_1^{(2)}+i\beta_0+i\beta_2}{4}\right)$$
$$=a_j^{'} \sinh\left(\frac{\alpha_1^{(1)}-\alpha_1^{(2)}+i\beta_0-i\beta_2}{4}\right)\exp\left(\frac{\alpha_1^{(1)}+\alpha_1^{(2)}+i\beta_0+i\beta_2}{4}\right). \quad (44)$$

Eliminating the nonzero factor $\exp\left(\frac{\alpha_1^{(1)}+\alpha_1^{(2)}+i\beta_0+i\beta_2}{4}\right)$, and expanding the hyperbolic sine functions yields that

$$(a_j+a_j^{'})\sinh\left(\frac{\alpha_1^{(2)}-\alpha_1^{(1)}}{4}\right)\cosh\left(\frac{i\beta_0-i\beta_2}{4}\right) \quad (45)$$
$$=(a_j-a_j^{'})\cosh\left(\frac{\alpha_1^{(2)}-\alpha_1^{(1)}}{4}\right)\sinh\left(\frac{i\beta_0-i\beta_2}{4}\right);$$

that is,

$$\frac{a_j+a_j^{'}}{a_j-a_j^{'}}\tanh\left(\frac{\alpha_1^{(2)}-\alpha_1^{(1)}}{4}\right)=\tanh\left[i\left(\frac{\beta_0-\beta_2}{4}\right)\right] \quad (46)$$

Since that $\tanh(i\chi)=i\tan\chi$, we have

$$\frac{a_j+a_j^{'}}{a_j-a_j^{'}}\tanh\left(\frac{\alpha_1^{(2)}-\alpha_1^{(1)}}{4}\right)=i\tanh\left(\frac{\beta_0-\beta_2}{4}\right) \quad (47)$$

Now we check the formula (47) carefully. Since $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, β_0, β_2 and a_j, a_j' are all real, the right hand side of (47) is pure imaginary while the left hand side is real. Thus, both vanish, or $\beta_2 = \beta_0$. In other words, it is impossible to obtain a new solution from the solution $i\beta_0$, $\alpha_1^{(1)}$, $\alpha_1^{(2)}$ by the formula (47). Therefore, the formulas (37) and (38) (the Backlund transformation in the references [3,4]) are trivial, by which no new solutions will be produced.

Topics relating to the Bäcklund transformation and nonlinear superposition formulas for nonlinear wave Equations are very active. And, in literature, there are some further discussion based on the formulas (37) and (38) (see [5-8,11,12], for instance). Therefore, the discussion in this note is, of course, necessary.

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