

Hopf Bifurcation in a Generalized Predator-Prey Model

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Abstract: A cubic system, which is a generalization of the predator-prey models, studied by many authors recently (see [1,2], for instance) is proposed. The properties of the equilibrium points, the Hopf bifurcation and the stability of the periodic solution, due to the bifurcation are investigated.

Keywords: cubic system; predator-prey; Hopf bifurcation; periodic solution.

Introduction

The study of limit cycles is an interesting topic in both mathematics and applied sciences. Normally it includes two aspects: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in the system (e.g. bifurcation). For the exact number of limit cycles and their relative positions, the known results are not many because determining the number and positions of limit cycles is not easy. That is the reason why the 16th Hilbert problem still remains open even for the case when $n = 2$ after one hundred years, although some important progress has been made recently [3-9].

The development of the qualitative analysis of ordinary differential equations is deriving not only by the "Hilbert Problems" proposed in the Second International Congress of Mathematicians, Paris 1900, but also by the study of the nonlinear oscillations in many other fields, such as discontinuous automatic control systems [6], bio-chemical reactions [10,11], immune response and predator-prey systems, and other problems in mathematical bi-sciences [12-15]. Qualitative analysis is

now a powerful tool in the study of nonlinear phenomena in all areas in science and technology, and it is developing very rapidly. In this paper, we study a cubic differential system which is a generalization of the predator-prey model studied recently by many authors [1,2,16,17]. We analyze the properties of the equilibrium points, and study the Hopf bifurcation and the stability of the periodic solution created by the bifurcation. This work is useful for a further understanding of the nonlinear oscillations of the predator-prey competition.

2. The Cubic Model and Main Theorems

We consider the system

$$\begin{aligned}\frac{dx}{dt} &= b_1x + b_2x^2 - b_3x^3 - b_4xy \\ \frac{dy}{dt} &= -cy + (\alpha x - \beta y)y,\end{aligned}\tag{1}$$

where

b_1 , nonnegative, $b_3, b_4, c, \alpha, \beta$, positive, and b_2 sign undetermined parameters.

The system (1) can be considered a special

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case of the following model for predator-prey interaction:

$$\begin{aligned} \frac{dx}{dt} &= f(x) - g(x, y) \\ \frac{dy}{dt} &= u(g(x, y), y) - v(y), \end{aligned} \tag{2}$$

where, x and y represent densities of prey and predator, respectively. The functions $f, g, u,$ and v represent the rates of prey reproduction, prey death due to predation, predator reproduction, and predator death, respectively. Gilpin (see Kuno [17], for instance) used a function of the form $f(x) = ax - bx^2 + cx^3$, in his predator-prey model, which can be described both over- and under-crowding effects in the prey population. And many Chinese authors ([1,2]) have used some other forms for $f(x)$ and other functions of (2).

By the variable transform:

$x = \frac{c}{\alpha} \bar{x}, y = \frac{c}{\beta} \bar{y}, dt = \frac{1}{c} d\tau,$ and then replace \bar{x}, \bar{y}, τ with $x, y, t,$ the system (1) is transferred to

$$\begin{aligned} \frac{dx}{dt} &= x(a_1 + a_2x - a_3x^2) - kxy \\ \frac{dy}{dt} &= y(-1 + x - y), \end{aligned} \tag{3}$$

where

$$a_1 = \frac{b_1}{c}, \text{ nonnegative } a_3 = \frac{b_3c}{\alpha^2}, k = \frac{b_4}{\beta},$$

positive, and $a_2 = \frac{b_2}{\alpha}.$

It is easy to see that, in $\Omega = \{(x, y) | x \geq 0, y \geq 0\},$ if $a_1 + a_2 \geq a_3$ the system has three equilibrium points: $O(0,0),$

$B(x_2, 0),$ and $E(x_4, x_4 - 1),$ where

$$\begin{aligned} x_2 &= \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3}, \\ x_4 &= \frac{(a_2 - k) + \sqrt{(a_2 - k)^2 + 4(a_1 + k)a_3}}{2a_3}. \end{aligned}$$

Let

$$\Delta = (a_2 - k)^2 + 4(a_1 + k)a_3 \tag{4}$$

we have

$$x_4 = \frac{a_2 - k + \sqrt{\Delta}}{2a_3}$$

Let $\Omega^+ = \{(x, y) | x > 0, y > 0\}.$ Consider the eigenvalues of the variational matrix of system (3), or of the Jacobian:

$$J(x, y) = \begin{pmatrix} a_1 + 2a_2x - 3a_3x^2 - ky & -kx \\ y & -1 + x - 2y \end{pmatrix}, \tag{5}$$

and, denote

$$\begin{aligned} p &= (-1 - a_2 + 2k)x_4 + (1 - 2a_1 - 2k), \\ R &= (2k - 1) + \frac{1 - 2a_1 - 2k}{x_4}. \end{aligned} \tag{6}$$

It follows that

- (1) $O(0,0)$ is always a saddle since the eigenvalues of $J(0,0) : a_1$ and $-1,$ having opposite signs;
- (2) $B(x_2, 0)$ is a saddle if $a_1 + a_2 > a_3$ since the eigenvalues: $-a_1 - a_3x_2^2$ and $x_2 - 1,$ having opposite signs; it is a stable node or focus if $a_1 + a_2 < a_3.$
- (3) $E(x_4, x_4 - 1)$ is a stable node or focus if $a_1 + a_2 > a_3$ and $p < 0,$ or it is an unsta-

ble node or focus if $a_1 + a_2 > a_3$ and $p < 0$.

Note that when $a_1 + a_2 > a_3$, E is the only equilibrium point in Ω^+ .

We prove the following theorems:

Theorem 1. System (3) undergoes a Hopf bifurcation at $a_2 = R$. The periodic solution created by the bifurcation is stable if $a_2 < 2k - 1$, and unstable if $a_2 > 2k - 1$.

Proof. Compute the Jacobian at $E(x_4, x_4 - 1)$ with the characteristic equation

$$\lambda^2 - p\lambda + q = 0 \quad (7)$$

where p is as defined in (6), and

$$q = (a_3x_4^2 + a_1 + k)(x_4 - 1) > 0 \quad (\text{since } a_1 + a_2 > a_3). \quad (8)$$

Choose p as the bifurcation parameter μ . It follows that

$$\begin{aligned} a_2 &= \frac{1 - 2a_1 - 2k - p}{x_4} + 2k - 1 \\ &= \frac{1 - 2a_1 - 2k - \mu}{x_4} + 2k - 1. \end{aligned} \quad (9)$$

We may rewrite system (3) as

$$\begin{aligned} \frac{dx}{dt} &= P(x, y, \mu), \\ \frac{dy}{dt} &= Q(x, y, \mu). \end{aligned} \quad (10)$$

Denote the Jacobian of (10) as $J(\mu)$, and then the trace of the matrix $J(\mu)$ is just $-p$, in other words, by the second equation in (9), we have

$$trJ(\mu) = -1 + 2a_1 + 2k + (1 + a_2 - 2k)x_4. \quad (11)$$

Since

$$\left. \frac{d}{d\mu} trJ(\mu) \right|_{\mu=0} = x_4 > 0$$

the function $trJ(\mu)$ is increasing at $\mu = 0$. It is easy to know, by (11) and (9), that $trJ(\mu)|_{\mu=0} = 0$, and we have

$$trJ(\mu) \begin{cases} < 0 & \text{if } \mu < 0 \\ = 0 & \text{if } \mu = 0 \\ > 0 & \text{if } \mu > 0. \end{cases} \quad (12)$$

Notice that $q > 0$, the sign of the real parts of the roots of the equation (7) is determined by $trJ(\mu)$ only, which is changed from negative to positive when μ is increasing from negative to positive. This means that the phase structure of $E(x_4, x_4 - 1)$ changes from stable to unstable at $\mu = 0$ as μ increases.

We still need to show that when $\mu = 0$, or, when $a_2 = (2k - 1) + \frac{1 - 2a_1 - 2k}{x_4}$, the equilibrium point $E(x_4, x_4 - 1)$ is a first order focus. Let

$$\begin{aligned} u &= y - y_4, \quad v = \frac{y_4}{\sqrt{q}}(y - x + y_4 - x_4) \\ &= \frac{y_4}{\sqrt{q}}(y - x - 1), \quad \tau = \sqrt{q}t, \end{aligned}$$

and transfer the system (3) to

$$\begin{aligned} \frac{du}{d\tau} &= -v - \frac{1}{y_4}uv \\ \frac{dv}{d\tau} &= u + A_1u^2 + A_2v^2 - A_3uv \\ &\quad + A_4u^3 - A_5v^3 - A_6u^2v + A_7uv^2, \end{aligned} \quad (13)$$

where

$$A_1 = \frac{y_4(3a_3x_4 - a_2 + k)}{q}, \quad A_2 = \frac{3a_3x_4 - a_2}{y_4},$$

$$A_3 = \frac{6a_3x_4 - 2a_2 + k + 1}{\sqrt{q}},$$

$$A_4 = \frac{a_3y_4}{q}, \quad A_5 = \frac{a_3\sqrt{q}}{y_4^2}, \quad A_6 = \frac{3a_3}{\sqrt{q}}, \quad A_7 = \frac{3a_3}{y_4}.$$

Using the polar system: $u = r \cos \theta, v = r \sin \theta$, the system (13) is now

$$\frac{dr}{dt} = \left(-\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta + A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta\right) r^2$$

$$+ (A_4 \sin \theta \cos^3 \theta - A_5 \sin^4 \theta - A_6 \sin^2 \theta \cos^2 \theta + A_7 \sin^3 \theta \cos \theta) r^3,$$

$$\frac{d\theta}{dt} = 1 + (A_1 \cos^3 \theta + A_2 \sin^2 \theta \cos \theta - A_3 \sin \theta \cos^2 \theta + \frac{1}{y_4} \sin^2 \theta \cos \theta) r$$

$$+ (A_4 \cos^4 \theta - A_5 \sin^3 \theta \cos \theta - A_6 \sin \theta \cos^3 \theta + A_7 \sin^2 \theta \cos^4 \theta) r^2.$$

Therefore,

$$\frac{dr}{d\theta} = \left(-\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta + A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta\right) r^2$$

$$+ \{(A_4 \sin \theta \cos^3 \theta - A_5 \sin^4 \theta - A_6 \sin^2 \theta \cos^2 \theta + A_7 \sin^3 \theta \cos \theta)$$

$$- [(-\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta + A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta)$$

$$(A_4 \cos^3 \theta + A_2 \sin^2 \theta \cos \theta - A_3 \sin \theta \cos^2 \theta + \frac{1}{y_4} \sin^2 \theta \cos \theta)]\} r^3 + \dots$$

(14)

Let the solution of (14) take the form:

$$r = c + r_2(\theta)c^2 + r_3(\theta)c^3 + \dots,$$

with $r_2(0) = r_3(0) = \dots = 0$. (15)

Substituting (15) into (14) and comparing the coefficients of c^2 , one has

$$\frac{dr_2}{d\theta} = -\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta$$

$$+ A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta \quad (16)$$

Taking the integration of (16) on $[0, \theta]$, one obtains

$$r_2(\theta) = \frac{1}{3} \left(\frac{1}{y_4} - A_1 + A_2\right) \cos^3 \theta - \frac{1}{3} A_3 \sin^3 \theta$$

$$- A_2 \cos \theta - \frac{1}{3} \left(\frac{1}{y_4} - A_1\right) + \frac{2}{3} A_2,$$

which is a periodic function of period 2π .

Similarly, for the coefficients of c^3 ,

$$\frac{dr_3}{d\theta} = (A_4 \sin \theta \cos^3 \theta - A_5 \sin^4 \theta - A_6 \sin^2 \theta \cos^2 \theta + A_7 \sin^3 \theta \cos \theta)$$

$$- \left(-\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta + A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta\right)$$

$$(A_4 \cos^3 \theta + A_2 \sin^2 \theta \cos \theta - A_3 \sin \theta \cos^2 \theta + \frac{1}{y_4} \sin^2 \theta \cos \theta)$$

$$+ \frac{2}{3} \left(-\frac{1}{y_4} \sin \theta \cos^2 \theta + A_1 \sin \theta \cos^2 \theta + A_2 \sin^3 \theta - A_3 \sin^2 \theta \cos \theta\right)$$

$$\left[\left(\frac{1}{y_4} - A_1 + A_2\right) \cos^3 \theta - A_3 \sin^3 \theta - 3A_2 \cos \theta - \left(\frac{1}{y_4} - A_1 - 2A_2\right)\right].$$

(17)

Let $r_3(\theta) = g_3\theta + f_3(\theta)$, then

$$g_3 = \frac{1}{2\pi} \int_0^{2\pi} \left[\left(-A_5 - \frac{2}{3} A_2 A_3\right) + (2A_4 + 6A_2 A_3 - A_6 - \frac{2}{3} A_1 A_3 + \frac{5}{3} \frac{A_3}{y_4}) \cos^2 \theta + \right.$$

$$\left. (-A_5 + A_6 - \frac{26}{3} A_2 A_3 + 4A_4 A_3 - 5 \frac{A_3}{y_4}) \cos^4 \theta + \right.$$

$$\left. \left(-\frac{10}{3} A_1 A_3 + \frac{10}{3} A_2 A_3 + \frac{10}{3} \frac{A_3}{y_4}\right) \cos^6 \theta \right] d\theta$$

$$= \frac{1}{8} (A_2 A_3 + A_1 A_3 - 3A_5 - A_6);$$

and

$$f_3(\theta) = \int_0^\theta \left\{ (A_4 - 2A_1 A_2 + \frac{2A_2}{y_4}) \sin \theta \cos^3 \theta + (A_7 - 2A_2^2) \sin^3 \theta \cos \theta \right.$$

$$\left. - (A_1 - \frac{1}{y_4}) \left(\frac{5}{3} A_1 - \frac{2}{3} A_2 - \frac{2}{3y_4}\right) \sin \theta \cos^5 \theta - \left[(A_1 - \frac{1}{y_4}) \left(A_2 + \frac{1}{y_4}\right) + A_1 A_2 \right. \right.$$

$$\left. + A_3^2 + \frac{2}{3} A_2 \left(\frac{1}{y_4} - A_1 + A_2\right)\right] \sin^3 \theta \cos^2 \theta - \left(A_2^2 - \frac{2}{3} A_3^2 + \frac{A_2}{y_4}\right) \sin^5 \theta \cos \theta$$

$$- \frac{2}{3} \left(A_1 - \frac{1}{y_4}\right) \left(\frac{1}{y_4} - A_1 - 2A_3\right) \sin \theta \cos^2 \theta - \frac{2}{3} A_2 \left(\frac{1}{y_4} - A_1 - 2A_2\right) \sin^3 \theta$$

$$\left. + \frac{2}{3} \left(\frac{A_3}{y_4} - A_1 A_3 - 2A_2 A_3\right) \cos \theta - \frac{2}{3} \left(\frac{A_3}{y_4} - A_1 A_3 - 2A_2 A_3\right) \cos^3 \theta \right\} d\theta.$$

It is easy to see that $f_3(\theta)$ is also a periodic function with the period 2π .

Notice that when $a_2 = R$, (or $p = 0$), $y_4 = a_2 x_4 - 2a_3 x_4^2$, thus

$$\begin{aligned} g_3 &= \frac{a_3 k x_4}{8q\sqrt{q}}(1 - 2a_1 - 2k) \\ &= \frac{a_3 k x_4}{8q\sqrt{q}}(a_2 + 1 - 2k) \neq 0. \end{aligned} \quad (18)$$

By the criteria of the successor function, $E(x_4, x_4 - 1)$ is a first order stable focus if $g_3 < 0$, or $a_2 < 2k - 1$, and unstable if $a_2 > 2k - 1$. Moreover, when it is unstable, by the method of Friedrich (see, for example, [17]), the periodic solution surrounding $E(x_4, x_4 - 1)$ is stable. We thus complete the proof of Theorem 1.

3. Applications to the Predator-prey Systems

We use an example to illustrate our theorems. Let $a_1 = 0$ in the system (3), one has

$$\begin{aligned} \frac{dx}{dt} &= x(a_2 x - a_3 x^2) - kxy \\ \frac{dy}{dt} &= y(-1 + x - y) \end{aligned} \quad (19)$$

which is studied by [1,2] recently. It is easy to see that (x^*, y^*) , where

$$x^* = \frac{(a_2 - k) + \sqrt{(a_2 - k)^2 + 4ka_3}}{2a_3} \quad (20)$$

$$y^* = x^* - 1,$$

is the only equilibrium point in Ω^+ if $a_2 > a_3$.

Assume

$$R_0 = (2k - 1) \frac{y^*}{x^*}.$$

Theorem 2. If $0 < a_2 < a_3$, the equilibrium $(a_2/a_3, 0)$ of the system (19) is globally asymptotically stable.

Proof. The Jacobian of system (19) at $(a_2/a_3, 0)$ is

$$J(x, y) \Big|_{(x,y)=(a_2/a_3,0)} = \begin{pmatrix} -2a_2^2/a_3 & -ka_2/a_3 \\ 0 & -1 + a_2/a_3 \end{pmatrix}.$$

Both of the eigenvalues are negative if $0 < a_2 < a_3$. Also, by the fact that all the trajectories of system (19) for $t > 0$ are bounded in Ω (see [2]), thus its ω -limit set contains only equilibrium points, close orbits, or singular close orbits. Note that both x -axis and y -axis are the orbits of (19), and there is no other equilibrium point in $\Omega^+ = \{(x, y) | x > 0, y > 0\}$. Therefore, all the trajectories approach to $(a_2/a_3, 0)$ for $t > 0$. $(a_2/a_3, 0)$ is globally asymptotically stable.

Note that if $a_2 = R_0 = (2k - 1) \frac{y^*}{x^*}$, then $a_2 < 2k - 1$. We have

Theorem 3. If $a_2 > a_3$, then (x^*, y^*) is the only equilibrium point in Ω^+ . The system (19) undergoes a Hopf bifurcation at

$R_0 = (2k - 1) \frac{y^*}{x^*}$, and the periodic solution of the system created by the bifurcation is always stable.

These theorems have not been studied in [1,2], and they are useful in the analysis of the nonlinear oscillatory behavior of the predator and prey populations.

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