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On a Batch Arrival Queue with Setup and Uncertain Parameter Patterns

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Abstract: This work constructs the membership functions of the system characteristics of a batch queuing system with server setup, in which the batch arrival rate, customer service rate, and server setup rate are all fuzzy numbers. The α-cut approach is used to transform a fuzzy queue into a family of conventional crisp queues in this context. By means of the membership functions of the system characteristics, a set of parametric nonlinear programs is developed to describe the family of crisp queues with server setup. Two numerical examples are solved successfully to illustrate the validity of the proposed approach. Because the system characteristics are expressed and governed by the membership functions, the fuzzy queues with server setup are represented more accurately and the analytic results are more useful for system designers and practitioners.

Keywords: Fuzzy sets; Membership function; Nonlinear programming; Setup

1. Introduction

Queueing models with a setup are effective methods for performance analysis of computer and telecommunication systems, manufacturing/production systems and inventory control (Borthakur and Medhi [1], Li et al. [2], Lee and Park [3], Krishna Reddy et al. [4], Choudhury ([5,6], and Ke [7]). In practical situations, the server setup corresponds to the preparatory work of the service before starting the service. For example, when a typical machine tool is turned on, a proper setup time before the normal use state is necessary for the machine tool. During this setup time, the lubricant can increasingly extend its function and the number of the unsteady conditions of the machine tool is reduced. Because the most dramatic changes of the thermal distortion occurs during the first thirty minutes of the start-up process, to maintain the precise state the machine tool without a proper setup time is installed and run in a controlled air-conditional room which has relatively high maintenance cost. On the other hand, the machine tool with the start-up process with a setup time period can operate in regular workspace not the controlled environment. Hence, the start-up process with the setup time is cost-effective.

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Li et al. [2] proposed an efficient iterative algorithm to compute the stationary queue length distributions for the M/G/1/N[1]queues with setup time and arbitrary state dependent arrival rate. Lee and Park [3] examined a like-queue M/G/1 production system with early setup and then developed a procedure to find the joint optimal thresholds that minimize a linear average cost. Krishna Reddy et al. [4] studied a N policy $M^{[x]}/G/1$ queueing system with multiple vacations and setup times for a two stage flow line production system.

In the literature described above, customer inter-arrival times, customer service times and server setup times are required to follow certain probability distributions with fixed parameters. However, in many real-world applications, the parameter distributions may only be characterized subjectively; that is, the arrival, service and setup patterns are typically described in everyday language summaries of central tendency, such as "the mean arrival rate is around 5 per day", "the mean service rate is about 10 per hour" or "the mean setup rate is approximately 2 per day", rather than with complete probability distributions. In other words, these system parameters are both possibilistic and probabilistic. Thus, fuzzy queues are potentially much more useful and realistic than the commonly used crisp queues (see Li and Lee [8] and Zadeh [9]). By extending the usual crisp queues to fuzzy queues in the context of a setup server, these queuing models become appropriate for a wider range of applications.

Li and Lee [8] investigated the analytical results for two typical fuzzy queues (denoted $M/F/1/\infty$ and $FM/FM/1/\infty$, where F represents fuzzy time and FM represents fuzzified exponential distributions) using a general approach based on Zadeh's extension principle (see also Prade [10] and Yager [11]), the possibility concept and fuzzy Markov chains (see Stanford [12]). A useful modeling and inferential technique would be to apply their approach to general fuzzy queuing problems. However, their approach is complicated and not suitable for computational purposes; moreover, it cannot easily be used to derive analytic results for other complicated queuing systems (see Negi and Lee [13]). In particular, it is very difficult to apply this approach to fuzzy queues with three or more fuzzy variables. Negi and Lee [13] proposed a procedure using α-cuts and two-variable simulation to analyze fuzzy queues (see also Chanas and Nowakowski [14]). Unfortunately, their approach provides only crisp solutions; *i.e.*, it does not fully describe the membership functions of the system characteristics. Using parametric programming, Kao et al. [15] constructed the membership functions of the system characteristics for fuzzy queues and successfully applied them to four simple fuzzy queue models: $M/F/1/\infty$, $F/M/1/\infty$, $F/F/1/\infty$ and $FM/FM/1/\infty$. Chen [16, 17] developed $FM/FM/1/L$ and $FM/FM^{[K]}/1/\infty$ fuzzy systems. Recently, Ke and Lin [18] and Ke et al. [19], who respectively analyzed fuzzy queues with unreliable-server and retrial systems using the same approach (Zadeh's extension principle).

All previous researches on fuzzy queuing models are focused on ordinary queues with one or two fuzzy variables. In this paper, we develop an approach that provides system characteristics for queues with a setup and three fuzzy variables: fuzzified exponential arrival, service and setup rates. Through α-cuts and Zadeh's extension principle, we transform the fuzzy queues to a family of crisp queues. As α varies, the family of crisp queues is described and solved using

[1] Kendall's shorthand notation a/b/c/d is widely used to describe queueing models. In this notation, a specifies the interarrival time (arrival process), b specifies the service time, c is the number of servers, and d is the restriction of system capacity. In many situations only the first three symbols are used. Current practice is to omit the service-capacity symbol if no restriction is imposed ($d = \infty$). The following symbols are used: M for the exponential distribution (for memoryless), G for a general distribution, D for deterministic, …, etc.

parametric nonlinear programming (NLP). The NLP solutions completely and successfully yield the membership functions of the system characteristics, including the expected number of customers in the system, the expected waiting time in the queue, and the expected lengths of time the server is idle and busy, Although an explicit closed-form expression for the membership function is very difficult to obtain in the case of three fuzzy variables, we develop a characterization that yields closed-form expressions when interval limits are invertible. Furthermore, this paper extends the analysis of system characteristics to encompass other system indices that are useful in more realistic systems.

The remainder of this paper is organized as follows. Section 2 presents the system characteristics of standard and fuzzy queuing models with a setup time. In Section 3, a mathematical programming approach is developed to derive the membership functions of these system characteristics. To demonstrate the validity of the proposed approach, two realistic numerical examples are described and solved. Discussion is provided in Section 4, and conclusions are drawn in Section 5. For notational convenience, our model in this paper is hereafter denoted $FM^{[x]}/FM/1/FSET$, where $FSET$ represents the fuzzified exponential setup rate.

2. Fuzzy queues with a setup time

2.1. M[x]/M/1/SET queues

We consider a batch arrival queuing system with a setup, where the server has a setup time before serving the first customer who initializes a busy period. It is assumed that customers arrive in batches to occur according to a compound Poisson process with arrival rate λ. Let A_k denote the number of customers belonging to the k^{th} arrival batch, where A_k, $k = 1, 2, 3, \cdots$, are with a common distribution

$$\Pr[A_k = n] = a_n, n = 1,\ 2,\ 3, \cdots \tag{1}$$

and $E[A] = \sum_{n=1}^{\infty} n a_n$. Customers arriving at the server form a single-file queue and are served in order. The service time is exponentially distributed with rate μ. The server can serve only one customer at a time. When one or more customers arrive, the server is immediately reactivated but is temporarily unavailable to the waiting customers. He needs a random setup time with an exponential rate θ before providing service. Once the setup is over, the server immediately starts serving the waiting customers until the system becomes empty. Define:

$N_s \equiv$ the expected number of customers in the system,

$W_q \equiv$ the expected waiting time in the queue,

$E[B] \equiv$ the expected length of time the server is busy,

$E[I] \equiv$ the expected length of time the server is idle.

From the results in Takagi [20], we can easily derive the system characteristics in terms of the system parameters:

$$N_s = \frac{(\lambda E[A])^2}{\mu(\mu - \lambda E[A])} + \frac{\lambda E[A(A-1)]}{2(\mu - \lambda E[A])} + \frac{\lambda(\mu + \theta)E[A]}{\mu\theta} \tag{2}$$

$$W_q = \frac{\lambda E[A]}{\mu(\mu - \lambda E[A])} + \frac{E[A(A-1)]}{2E[A](\mu - \lambda E[A])} + \frac{1}{\theta} \tag{3}$$

$$E[B] = \frac{\theta + \lambda E[A]}{\theta(\mu - \lambda E[A])} \tag{4}$$

$$E[I] = \frac{\lambda E[A] + \theta}{\lambda \theta E[A]} \tag{5}$$

In steady-state, it is necessary that we have

$$0 < \frac{\theta(\mu - \lambda)}{\mu(\lambda + \theta)} < 1.$$

2.2. FM[x]/FM/1/FSET queues

To extend the applicability of the standard queuing model with a setup time, we allow for fuzzy specification of system parameters. Suppose the arrival rate λ, service rate μ and setup rate θ are approximately known and can be represented by the fuzzy sets $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\theta}$, respectively. Let $\phi_{\tilde{\lambda}}(x)$, $\phi_{\tilde{\mu}}(y)$ and $\phi_{\tilde{\theta}}(v)$ denote the membership functions of $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\theta}$, respectively. We then have the following fuzzy sets:

$$\tilde{\lambda} = \left\{ (x, \phi_{\tilde{\lambda}}(x)) \,\middle|\, x \in X \right\} \tag{6a}$$

$$\tilde{\mu} = \left\{ (y, \phi_{\tilde{\mu}}(y)) \,\middle|\, y \in Y \right\} \tag{6b}$$

$$\tilde{\theta} = \left\{ (v, \phi_{\tilde{\theta}}(v)) \,\middle|\, v \in V \right\} \tag{6c}$$

where X, Y and V are the crisp universal sets of the arrival, service and setup rates, respectively.

Let $f(x, y, v)$ denote the system characteristic of interest. Since $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\theta}$ are fuzzy numbers, $f(\tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ is also a fuzzy number. Following Zadeh's extension principle (see Zadeh [9] and Yager [11]), the membership function of the system characteristic $f(\tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ is defined as:

$$\phi_{f(\tilde{\lambda}, \tilde{\mu}, \tilde{\theta})}(z) = \sup_{\Omega} \min \left\{ \phi_{\tilde{\lambda}}(x), \phi_{\tilde{\mu}}(y), \phi_{\tilde{\theta}}(v) \,\middle|\, z = f(x, y, v) \right\} \tag{7}$$

where the supremum is taken over the set

$$\Omega = \left\{ x \in X, y \in Y, v \in V \,\middle|\, 0 < \frac{v(y - x)}{y(x + v)} < 1 \right\}.$$

Assume that the system characteristic of interest is the expected number of customers in the system. It follows from Equation(2) that the expected number of customers in the system is:

$$f(x, y, v) = \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y + v)E[A]}{yv} \tag{8}$$

The membership function for the expected number of customers in the system is:

$$\phi_{\tilde{N}_s}(z) = \sup_{\Omega} \min \left\{ \begin{array}{l} \phi_{\tilde{\lambda}}(x), \phi_{\tilde{\mu}}(y), \phi_{\tilde{\theta}}(v) \,\middle|\, z \\ = \dfrac{(xE[A])^2}{y(y - xE[A])} + \dfrac{xE[A(A-1)]}{2(y - xE[A])} + \dfrac{x(y + v)E[A]}{yv} \end{array} \right\}. \tag{9}$$

Unfortunately, the membership function is not expressed in the usual forms, making it very difficult to imagine their shapes. In this paper we approach the representation problem using a mathematical programming technique. Parametric NLPs are developed to find the α-cuts of $f(\tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ based on the extension principle.

3. Parametric Nonlinear Programming

To re-express the membership function $\phi_{\tilde{N}_s}(z)$ of $\tilde{N}_s$ in an understandable and usable form, we adopt Zadeh's approach, which relies on α-cuts of $\tilde{N}_s$. Definitions for the α-cuts of $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{\theta}$ as crisp intervals are as follows:

$$\lambda(\alpha) = \left[x_\alpha^L, x_\alpha^U\right]$$
$$= \left[\min_{x\in X}\left\{x \mid \phi_{\tilde{\lambda}}(x) \ge \alpha\right\}, \max_{x\in X}\left\{x \mid \phi_{\tilde{\lambda}}(x) \ge \alpha\right\}\right] \quad (10a)$$

$$\mu(\alpha) = \left[y_\alpha^L, y_\alpha^U\right]$$
$$= \left[\min_{y\in Y}\left\{y \mid \phi_{\tilde{\mu}}(y) \ge \alpha\right\}, \max_{y\in Y}\left\{y \mid \phi_{\tilde{\mu}}(y) \ge \alpha\right\}\right] \quad (10b)$$

$$\theta(\alpha) = \left[v_\alpha^L, v_\alpha^U\right]$$
$$= \left[\min_{v\in V}\left\{v \mid \phi_{\tilde{\theta}}(v) \ge \alpha\right\}, \max_{v\in V}\left\{v \mid \phi_{\tilde{\theta}}(v) \ge \alpha\right\}\right] \quad (10c)$$

The constant arrival, service and setup rates are shown as intervals when the membership functions are no less than a given possibility level for α. As a result, the bounds of these intervals can be described as functions of α and can be obtained as: $x_\alpha^L = \min \phi_{\tilde{\lambda}}^{-1}(\alpha)$, $x_\alpha^U = \max \phi_{\tilde{\lambda}}^{-1}(\alpha)$, $y_\alpha^L = \min \phi_{\tilde{\mu}}^{-1}(\alpha)$, $y_\alpha^U = \max \phi_{\tilde{\mu}}^{-1}(\alpha)$, $v_\alpha^L = \min \phi_{\tilde{\theta}}^{-1}(\alpha)$ and $v_\alpha^U = \max \phi_{\tilde{\theta}}^{-1}(\alpha)$. Therefore, we can use the α-cuts of $\tilde{N}_s$ to construct its membership function since the membership function defined in Equation(9) is parameterized by α.

Using Zadeh's extension principle, $\phi_{\tilde{N}_s}(z)$ is the minimum of $\phi_{\tilde{\lambda}}(x)$, $\phi_{\tilde{\mu}}(y)$ and $\phi_{\tilde{\theta}}(v)$. To derive the membership function $\phi_{\tilde{N}_s}(z)$, we need at least one of the following cases to hold such that

$$z = \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv}$$

satisfies $\phi_{\tilde{N}_s}(z) = \alpha$:

Case (i): ($\phi_{\tilde{\lambda}}(x) = \alpha$, $\phi_{\tilde{\mu}}(y) \ge \alpha$, $\phi_{\tilde{\theta}}(v) \ge \alpha$)

Case (ii): ($\phi_{\tilde{\lambda}}(x) \ge \alpha$, $\phi_{\tilde{\mu}}(y) = \alpha$, $\phi_{\tilde{\theta}}(v) \ge \alpha$)

Case (iii): ($\phi_{\tilde{\lambda}}(x) \ge \alpha$, $\phi_{\tilde{\mu}}(y) \ge \alpha$, $\phi_{\tilde{\theta}}(v) = \alpha$)

This can be accomplished using parametric NLP techniques. The NLP to find the lower and upper bounds of the α-cut of $\phi_{\tilde{N}_s}(z)$ for Case (i) are:

$$(N_s)_\alpha^{L_1} = \min \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11a)$$

$$(N_s)_\alpha^{U_1} = \max \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11b)$$

for Case (ii) are:

$$(N_s)_\alpha^{L_2} = \min \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11c)$$

$$(N_s)_\alpha^{U_2} = \max \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11d)$$

and for Case (iii) are:

$$(N_s)_\alpha^{L_3} = \min \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11e)$$

$$(N_s)_\alpha^{U_3} = \max \frac{(xE[A])^2}{y(y - xE[A])} + \frac{xE[A(A-1)]}{2(y - xE[A])} + \frac{x(y+v)E[A]}{yv} \quad (11f)$$

From the definitions of $\lambda(\alpha)$, $\mu(\alpha)$ and $\theta(\alpha)$ in Equation (10), $x \in \lambda(\alpha)$, $y \in \mu(\alpha)$

and $v \in \theta(\alpha)$ can be replaced by $x \in [x_\alpha^L, x_\alpha^U]$, $y \in [y_\alpha^L, y_\alpha^U]$ and $v \in [v_\alpha^L, v_\alpha^U]$, respectively. The α-cuts form a nested structure with respect to α (see Kaufmann [21] and Zimmermann [22]); *i.e.*, given $0 \le \alpha_2 < \alpha_1 \le 1$, we have $[x_{\alpha_1}^L, x_{\alpha_1}^U] \subseteq [x_{\alpha_2}^L, x_{\alpha_2}^U]$, $[y_{\alpha_1}^L, y_{\alpha_1}^U] \subseteq [y_{\alpha_2}^L, y_{\alpha_2}^U]$ and $[v_{\alpha_1}^L, v_{\alpha_1}^U] \subseteq [v_{\alpha_2}^L, v_{\alpha_2}^U]$. Therefore, Equations (11a), (11c) and (11e) have the same smallest element and Equations (11b), (11d) and (11f) have the same largest element. To find the membership function $\phi_{\tilde{N}_s}(z)$, it suffices to find the left and right shape functions of $\phi_{\tilde{N}_s}(z)$, which is equivalent to finding the lower bound $(N_s)_\alpha^L$ and upper bound $(N_s)_\alpha^U$ of the α-cuts of $\tilde{N}_s$, which can be rewritten as:

$$(N_s)_\alpha^L = \min \frac{(xE[A])^2}{y(y-xE[A])} + \frac{xE[A(A-1)]}{2(y-xE[A])} + \frac{x(y+v)E[A]}{yv}$$

s.t. $x_\alpha^L \le x \le x_\alpha^U$, $y_\alpha^L \le y \le y_\alpha^U$ and $v_\alpha^L \le v \le v_\alpha^U$ (12a)

$$(N_s)_\alpha^U = \max \frac{(xE[A])^2}{y(y-xE[A])} + \frac{xE[A(A-1)]}{2(y-xE[A])} + \frac{x(y+v)E[A]}{yv}$$

s.t. $x_\alpha^L \le x \le x_\alpha^U$, $y_\alpha^L \le y \le y_\alpha^U$ and $v_\alpha^L \le v \le v_\alpha^U$ (12b)

At least one of x, y or v must hit the boundaries of their α-cuts to satisfy $\phi_{\tilde{N}_s}(z) = \alpha$. This model is a set of mathematical programs with boundary constraints and lends itself to the systematic study of how the optimal solutions change with x_α^L, x_α^U, y_α^L, y_α^U, v_α^L and v_α^U as α varies over [0,1]. The model is a special case of parametric NLPs (see Gal [23]).

The crisp interval $[(N_s)_\alpha^L, (N_s)_\alpha^U]$ obtained from Equation (12) represents the α-cuts of $\tilde{N}_s$. Again, by applying the results of Kaufmann [21] and Zimmermann [22] and convexity properties to $\tilde{N}_s$, we have $(N_s)_{\alpha_1}^L \ge (N_s)_{\alpha_2}^L$ and $(N_s)_{\alpha_1}^U \le (N_s)_{\alpha_2}^U$, where $0 < \alpha_2 < \alpha_1 \le 1$. In other words, $(N_s)_\alpha^L$ increases and $(N_s)_\alpha^U$ decreases as α increases. Consequently, the membership function $\phi_{\tilde{N}_s}(z)$ can be found from Equation (12).

If both $(N_s)_\alpha^L$ and $(N_s)_\alpha^U$ in Equation (12) are invertible with respect to α, then a left shape function $L(z) = [(N_s)_\alpha^L]^{-1}$ and a right shape function $R(z) = [(N_s)_\alpha^U]^{-1}$ can be derived, from which the membership function $\phi_{\tilde{N}_s}(z)$ is constructed:

$$\phi_{\tilde{N}_s}(z) = \begin{cases} L(z), & (N_s)_{\alpha=0}^L \le z \le (N_s)_{\alpha=1}^L, \\ 1, & (N_s)_{\alpha=1}^L \le z \le (N_s)_{\alpha=1}^U, \\ R(z), & (N_s)_{\alpha=1}^U \le z \le (N_s)_{\alpha=0}^U. \end{cases} \tag{13}$$

In most cases, the values of $(N_s)_\alpha^L$ and $(N_s)_\alpha^U$ cannot be solved analytically. Consequently, a closed-form membership function for $\phi_{\tilde{N}_s}(z)$ cannot be obtained. However, the numerical solutions for $(N_s)_\alpha^L$ and $(N_s)_\alpha^U$ at different possibility levels can be collected to approximate the shapes of $L(z)$ and $R(z)$. That is, the set of intervals $\{[(N_s)_\alpha^L, (N_s)_\alpha^U] \mid \alpha \in [0,1]\}$ shows the shape of $\phi_{\tilde{N}_s}(z)$, although the exact function is not known explicitly.

Note that the membership functions for the expected waiting time in the queue, the expected length of time the server is busy and idle can be expressed in a similar manner.

Because the system characteristics are described by membership functions, the values completely preserve all the fuzziness of the arrival rate, service rate and setup rate. However, managers or designers may prefer a single crisp value for a system characteristic rather than a fuzzy set. To address this demand, the fuzzy values of system characteristics are defuzzified using Yager's [22] ranking index method. Because Yager's method possesses the property of area compensation, this method is adopted to transform fuzzy values of system characteristics into crisp ones. Suitable values of system characteristics are calculated as

$$\mathrm{O}(E[\Lambda]) = \int_0^1 \frac{(E[\Lambda])_\alpha^L + (E[\Lambda])_\alpha^U}{2}\, d\alpha$$

where $E[\Lambda]$ is a convex fuzzy number, and $((E[\Lambda])_\alpha^L, (E[\Lambda])_\alpha^U)$ is the α-cut of $E[\Lambda]$. Note that this method is a robust ranking technique that possesses the properties of compensation, linearity, and additivity (see Fortemps & Roubens [25]).

4. Numerical example

This section we present two examples motivated by real-life systems to demonstrate the practical use of the proposed approach.

Example 1. A factory produces diamond-cylinder parts with milling machine. The number of arriving materials each time follows a geometric distribution with parameter $p = 0.5$; *i.e.*, the size of arriving materials A is $\Pr(A = k) = 0.5(1-0.5)^{k-1}$, $k = 1, 2, \cdots, \infty$. As soon as no materials arrive, the milling machine is turned off. When one or more materials arrive, the milling machine is immediately turned on. A proper setup time is performed before the normal use state for the milling machine (Because the most dramatic changes of the thermal distortion occurs during the first few minutes of the start-up process). Clearly, this problem can be described by FM/FM/1/FSET system. For efficiency, the management wants to get the system characteristics such as the expected number of materials in the system, the expected waiting time in the queue, the expected length of time the server is busy and idle.

The fuzzy expected number of materials in the system ($\tilde{N}_s$)

Suppose the arrival rate, service rate and setup rate are trapezoidal fuzzy numbers represented by $\tilde{\lambda} = [5, 6, 7, 8]$, $\tilde{\mu} = [23, 24, 25, 26]$ and $\tilde{\theta} = [1, 2, 3, 4]$. First, it is easy to find that $[x_\alpha^L, x_\alpha^U] = [5+\alpha, 8-\alpha]$, $[y_\alpha^L, y_\alpha^U] = [23+\alpha, 26-\alpha]$ and $[v_\alpha^L, v_\alpha^U] = [1+\alpha, 4-\alpha]$. Next, it is obvious that when $x = x_\alpha^U$, $y = y_\alpha^L$ and $v = v_\alpha^L$, the expected number of materials in the system attains its maximum value, and when $x = x_\alpha^L$, $y = y_\alpha^U$ and $v = v_\alpha^U$, the expected number of materials in the system attains its minimum value. According to Equation (12), the α-cuts of $\tilde{N}_s$ are:

$$(N_s)_\alpha^L = \frac{240 - 2\alpha - 10\alpha^2}{64 - 28\alpha + 3\alpha^2} \tag{14a}$$

$$(N_s)_\alpha^U = \frac{144 + 62\alpha - 10\alpha^2}{7 + 10\alpha + 3\alpha^2} \tag{14b}$$

The inverse functions of $(N_s)_\alpha^L$ and $(N_s)_\alpha^U$ exist, yielding the membership function:

$$\phi_{\tilde{N}_s}(z) = \begin{cases} \dfrac{14z - 1 - \sqrt{4z^2 + 52z + 2401}}{3z + 10}, & \dfrac{15}{4} \le z \le \dfrac{228}{39} \\ 1, & \dfrac{228}{39} \le z \le \dfrac{49}{5} \\ \dfrac{-5z + 31 + \sqrt{4z^2 + 52z + 2401}}{3z + 10}, & \dfrac{49}{5} \le z \le \dfrac{144}{7} \end{cases} \tag{15}$$

as shown in Figure 1. The overall shape turns out as expected.

Next, we perform α-cuts of arrival, service and setup rates and fuzzy expected number of materials in the system at eleven distinct α values: 0, 0.1, …, 1. Crisp intervals for fuzzy expected number of materials in the system at different possibilistic α levels are presented in Table 1. The fuzzy expected number of materials in the system $\tilde{N}_s$ has two characteristics to be noted. First, the support of $\tilde{N}_s$ ranges from 3.75 to 20.5714; this indicates that, though the expected number of materials in the system is fuzzy, it is impossible for its values to fall below 3.75 or exceed 20.5714. Second, the α-cut at $\alpha = 1$ contains the values from 5.8462 to 9.8, which are the most possible values for the fuzzy expected number of materials in the system.

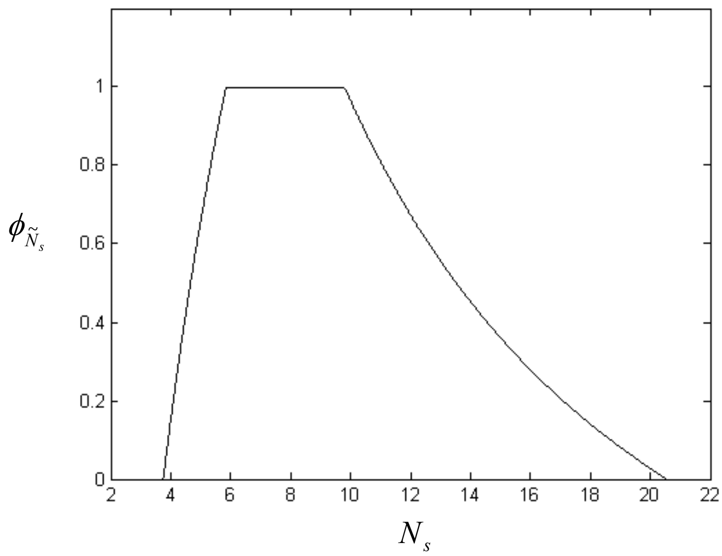

Figure 1. The membership function for fuzzy expected number of materials in the system

Table 1. α-cuts of arrival, service and setup rates and expected number of materials in the system

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(N_s)_\alpha^L$	$(N_s)_\alpha^U$
0.00	5.00	8.00	23.00	26.00	1.00	4.00	3.7500	20.5714
0.10	5.10	7.90	23.10	25.90	1.10	3.90	3.9147	18.6924
0.20	5.20	7.80	23.20	25.80	1.20	3.80	4.0875	17.1053
0.30	5.30	7.70	23.30	25.70	1.30	3.70	4.2688	15.7449
0.40	5.40	7.60	23.40	25.60	1.40	3.60	4.4595	14.5645
0.50	5.50	7.50	23.50	25.50	1.50	3.50	4.6601	13.5294
0.60	5.60	7.40	23.60	25.40	1.60	3.40	4.8716	12.6136
0.70	5.70	7.30	23.70	25.30	1.70	3.30	5.0948	11.7970
0.80	5.80	7.20	23.80	25.20	1.80	3.20	5.3309	11.0638
0.90	5.90	7.10	23.90	25.10	1.90	3.10	5.5809	10.4015
1.00	6.00	7.00	24.00	25.00	2.00	3.00	5.8462	9.8000

The fuzzy expected waiting time in the queue ($\tilde{W}_q$)

Using the same argument and Equation (3), the α-cuts of $\tilde{W}_q$ are:

$$(W_q)_\alpha^L = \frac{560-126\alpha+2\alpha^2}{1664-792\alpha+106\alpha^2-3\alpha^3} \quad (16a)$$

$$(W_q)_\alpha^U = \frac{200+114\alpha+2\alpha^2}{161+237\alpha+79\alpha^2+3\alpha^3} \quad (16b)$$

With the help of MATLAB® 7.0.4, the membership function is:

$$\phi_{\tilde{W}_q}(z) = \begin{cases} L(z), & \frac{35}{104} \le z \le \frac{436}{975} \\ 1, & \frac{436}{975} \le z \le \frac{79}{120} \\ R(z), & \frac{79}{120} \le z \le \frac{200}{161} \end{cases} \quad (17)$$

where:

$$R(z) = \frac{Q^{\frac{2}{3}} - (79z-2)Q^{\frac{1}{3}} + 2(2054z^2+355z+2)}{9zQ^{\frac{1}{3}}} \quad (18b)$$

with:

$$P = 259840z^3 + 66234z^2 - 2130z - 8 + 18z\sqrt{-5581488z^4 - 4705800z^3 - 9675963z^2 - 2204400z - 8547}, \quad (19a)$$

$$Q = -259840z^3 - 66234z^2 + 2130z + 8 + 18z\sqrt{-5581488z^4 - 4705800z^3 - 9675963z^2 - 2204400z - 8547}, \quad (19b)$$

as shown in Figure 2. The membership functions $L(z)$ and $R(z)$ have complex values with their imaginary parts approaching zero when $\frac{35}{104} \le z \le \frac{436}{975}$ for $L(z)$ and $\frac{79}{120} \le z \le \frac{200}{161}$ for $R(z)$. Hence, the imaginary parts of these two functions have no influence on the computational results and can be disregarded. Crisp intervals for the fuzzy expected waiting time in the queue at different possibilistic α levels are given in Table 2. For the fuzzy expected waiting time $\tilde{W}_q$, the range of $\tilde{W}_q$ at $\alpha = 1$ is [0.4472, 0.6583], indicating that it is definitely possible that expected waiting time falls between 0.4472 and 0.6583. Moreover, the range of $\tilde{W}_q$ at $\alpha = 0$ is [0.3365, 1.2422], indicating that the expected waiting time in the queue will never exceed 1.2422 or fall below 0.3365.

The fuzzy expected length of time the server is busy ($\tilde{E}[B]$)

Using the same argument and Equation (4), the α-cuts of $\tilde{E}[B]$ are:

$$(E[B])_\alpha^L = \frac{14+\alpha}{64-28\alpha+3\alpha^2} \quad (20a)$$

$$(E[B])_\alpha^U = \frac{17-\alpha}{7+10\alpha+3\alpha^2} \quad (20b)$$

The membership function is:

$$\phi_{\tilde{E}[B]}(z) = \begin{cases} \frac{28z+1-\sqrt{16z^2+224z+1}}{6z}, & \frac{7}{32} \le z \le \frac{15}{39} \\ 1, & \frac{15}{39} \le z \le \frac{4}{5} \\ \frac{-10z-1+\sqrt{16z^2+224z+1}}{6z}, & \frac{4}{5} \le z \le \frac{17}{7} \end{cases} \quad (21)$$

as shown in Figure 3. Crisp intervals for the fuzzy expected length of time server is busy at different possibilistic α levels are given in Table 3.

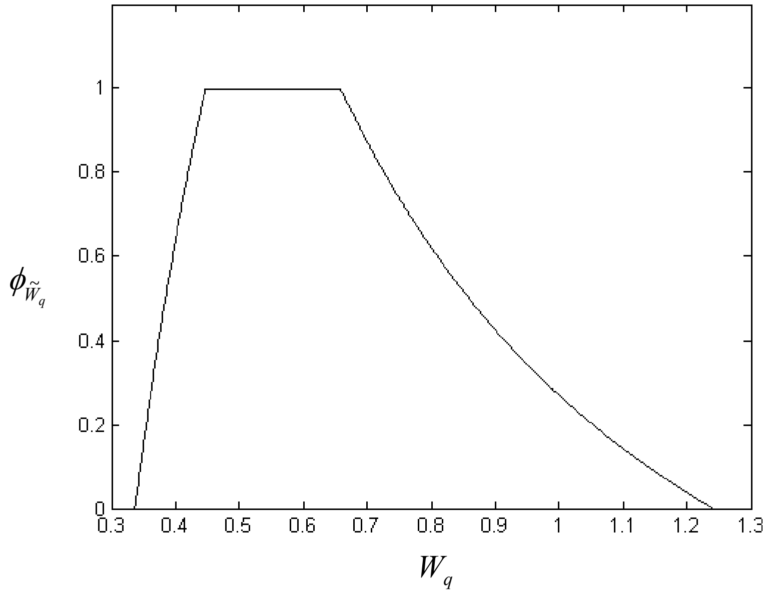

Figure 2. The membership function for fuzzy expected waiting time in the queue

Table 2. α-cuts of arrival, service and setup rates and expected waiting time in the queue

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(W_q)_\alpha^L$	$(W_q)_\alpha^U$
0.00	5.00	8.00	23.00	26.00	1.00	4.00	0.3365	1.2422
0.10	5.10	7.90	23.10	25.90	1.10	3.90	0.3452	1.1398
0.20	5.20	7.80	23.20	25.80	1.20	3.80	0.3543	1.0534
0.30	5.30	7.70	23.30	25.70	1.30	3.70	0.3638	0.9795
0.40	5.40	7.60	23.40	25.60	1.40	3.60	0.3739	0.9155
0.50	5.50	7.50	23.50	25.50	1.50	3.50	0.3844	0.8594
0.60	5.60	7.40	23.60	25.40	1.60	3.40	0.3956	0.8099
0.70	5.70	7.30	23.70	25.30	1.70	3.30	0.4074	0.7658
0.80	5.80	7.20	23.80	25.20	1.80	3.20	0.4199	0.7263
0.90	5.90	7.10	23.90	25.10	1.90	3.10	0.4331	0.6907
1.00	6.00	7.00	24.00	25.00	2.00	3.00	0.4472	0.6583

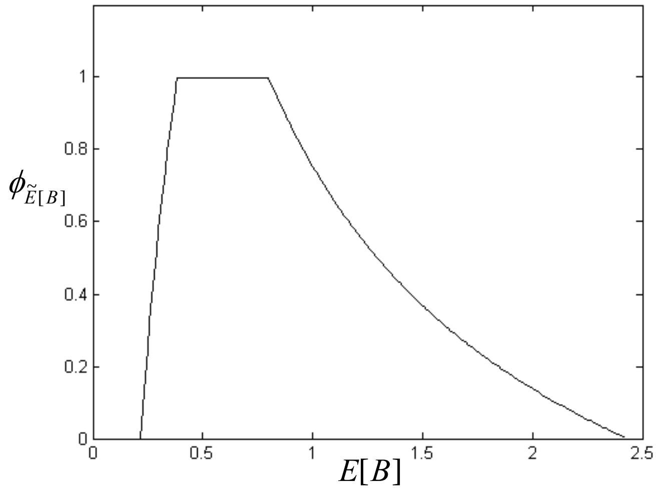

Figure 3. The membership function for fuzzy expected length of time the server is busy

Table 3. α -cuts of arrival, service and setup rates and expected length of time the server is busy

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(E[B])_\alpha^L$	$(E[B])_\alpha^U$
0.00	5.00	8.00	23.00	26.00	1.00	4.00	0.2188	2.4286
0.10	5.10	7.90	23.10	25.90	1.10	3.90	0.2303	2.1046
0.20	5.20	7.80	23.20	25.80	1.20	3.80	0.2427	1.8421
0.30	5.30	7.70	23.30	25.70	1.30	3.70	0.2560	1.6261
0.40	5.40	7.60	23.40	25.60	1.40	3.60	0.2703	1.4460
0.50	5.50	7.50	23.50	25.50	1.50	3.50	0.2857	1.2941
0.60	5.60	7.40	23.60	25.40	1.60	3.40	0.3024	1.1648
0.70	5.70	7.30	23.70	25.30	1.70	3.30	0.3205	1.0537
0.80	5.80	7.20	23.80	25.20	1.80	3.20	0.3401	0.9574
0.90	5.90	7.10	23.90	25.10	1.90	3.10	0.3614	0.8736
1.00	6.00	7.00	24.00	25.00	2.00	3.00	0.3846	0.8000

The fuzzy expected length of time the server is idle $(\tilde{E}[I])$

Using the same argument and Equation (5), the α -cuts of $\tilde{E}[I]$ are:

$$(E[I])_\alpha^L = \frac{20-3\alpha}{64-24\alpha+2\alpha^2} \tag{22a}$$

$$(E[I])_\alpha^U = \frac{11+3\alpha}{10+12\alpha+2\alpha^2} \tag{22b}$$

The membership function is:

$$\phi_{\tilde{E}[I]}(z) = \begin{cases} \dfrac{24z-3-\sqrt{64z^2+16z+9}}{4z}, & \dfrac{5}{16} \le z \le \dfrac{17}{42} \\ 1, & \dfrac{17}{42} \le z \le \dfrac{7}{12} \\ \dfrac{-12z+3+\sqrt{64z^2+16z+9}}{4z}, & \dfrac{7}{12} \le z \le \dfrac{11}{10} \end{cases} \tag{23}$$

as shown in Figure 4. Crisp intervals for the fuzzy expected length of time server is idle at different possibilistic α levels are provided in Table 4. From Figures 3-4 and Tables 3-4, we can gain insight into the possible expected time server is busy and idle.

Example 2. The positive film maker can export the images in a computer into films directly. To reduce the possible failure of the laser beam in the film maker, the correct operating steps of this film maker are to turn-on and to wait for setup time. After the setup time, the computer linked with the film maker is started for the continuous steps (This positive film maker can be viewed as a server). The number of arriving images each time follows a geometric distribution with parameter $p = 0.5$. Suppose the arrival rate, service rate and setup rate are trapezoidal fuzzy numbers represented by $\tilde{\lambda} = [5, 6, 7, 8]$, $\tilde{\mu} = [38,39,40,41]$ and $\tilde{\theta} = [1, 8, 9, 16]$. For efficiency, the designer wants to get the system characteristics such as the expected number of images in the system, the expected waiting time in the queue, the expected length of time the server is busy and idle and etc.

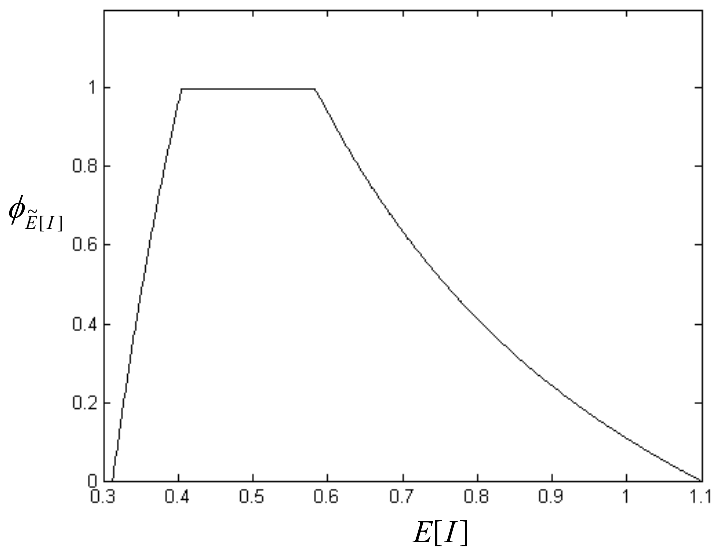

Figure 4. The membership function for fuzzy expected length of time the server is idle

Table 4. α-cuts of arrival and setup rates and expected length of time the server is idle

α	x_α^L	x_α^U	v_α^L	v_α^U	$(E[I])_\alpha^L$	$(E[I])_\alpha^U$
0.00	5.00	8.00	1.00	4.00	0.3125	1.1000
0.10	5.10	7.90	1.10	3.90	0.3197	1.0071
0.20	5.20	7.80	1.20	3.80	0.3273	0.9295
0.30	5.30	7.70	1.30	3.70	0.3352	0.8636
0.40	5.40	7.60	1.40	3.60	0.3436	0.8069
0.50	5.50	7.50	1.50	3.50	0.3524	0.7576
0.60	5.60	7.40	1.60	3.40	0.3617	0.7143
0.70	5.70	7.30	1.70	3.30	0.3715	0.6760
0.80	5.80	7.20	1.80	3.20	0.3819	0.6418
0.90	5.90	7.10	1.90	3.10	0.3930	0.6111
1.00	6.00	7.00	2.00	3.00	0.4048	0.5833

Following the solution procedure illustrated above, we obtain membership functions as shown in Figures 5 to 8 and crisp intervals for the expected number of images in the system, the expected waiting time in the queue and the expected lengths of time the server is busy and idle for different possibilistic α levels in Tables 5 to 8. We gain insight into the possible system characteristics.

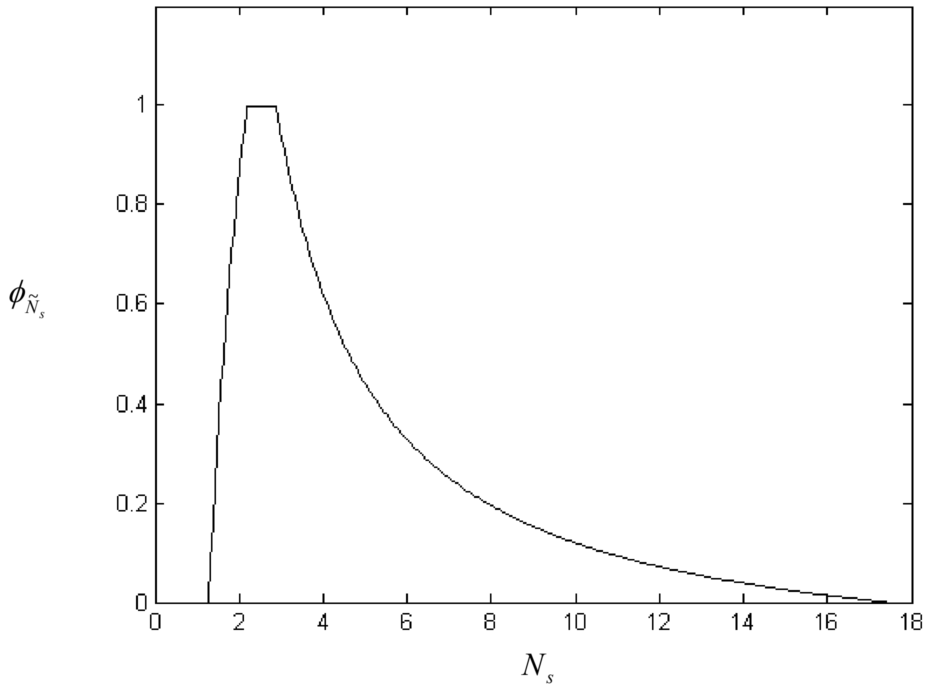

Figure 5. The membership function for fuzzy expected number of images in the system

Table 5. α-cuts of arrival, service and setup rates and expected number of images in the system

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(N_s)_\alpha^L$	$(N_s)_\alpha^U$
0.00	5.00	8.00	38.00	41.00	1.00	16.00	1.2702	17.4545
0.10	5.10	7.90	38.10	40.90	1.70	15.30	1.3312	10.7112
0.20	5.20	7.80	38.20	40.80	2.40	14.60	1.3965	7.8805
0.30	5.30	7.70	38.30	40.70	3.10	13.90	1.4669	6.3127
0.40	5.40	7.60	38.40	40.60	3.80	13.20	1.5430	5.3103
0.50	5.50	7.50	38.50	40.50	4.50	12.50	1.6258	4.6099
0.60	5.60	7.40	38.60	40.40	5.20	11.80	1.7163	4.0899
0.70	5.70	7.30	38.70	40.30	5.90	11.10	1.8160	3.6862
0.80	5.80	7.20	38.80	40.20	6.60	10.40	1.9266	3.3621
0.90	5.90	7.10	38.90	40.10	7.30	9.70	2.0504	3.0950
1.00	6.00	7.00	39.00	40.00	8.00	9.00	2.1905	2.8700

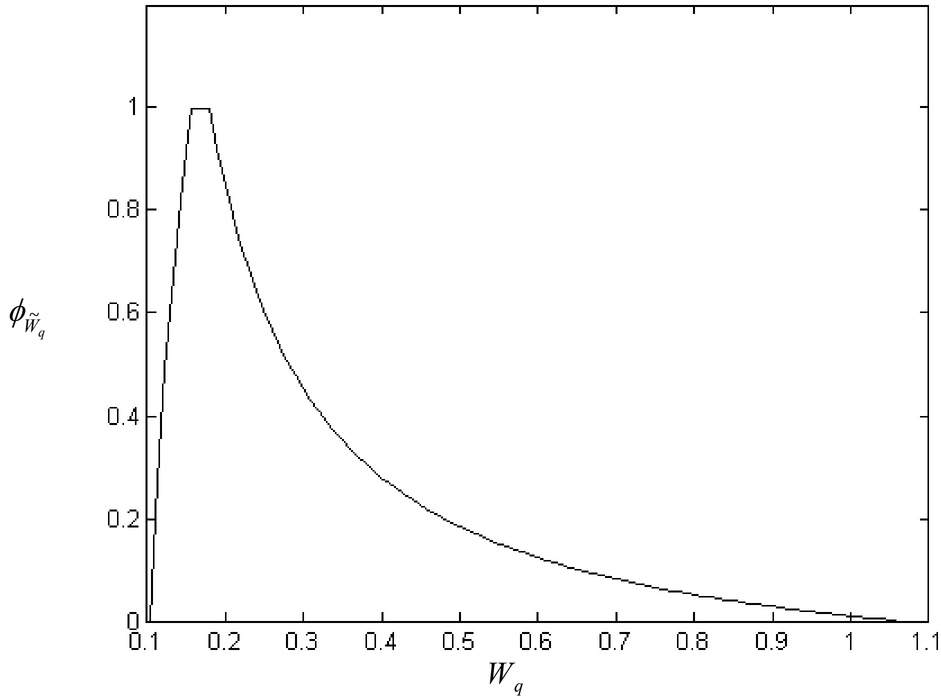

Figure 6. The membership function for fuzzy expected waiting time in the queue

Table 6. α-cuts of arrival, service and setup rates and expected waiting time in the queue

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(W_q)_\alpha^L$	$(W_q)_\alpha^U$
0.00	5.00	8.00	38.00	41.00	1.00	16.00	0.1026	1.0646
0.10	5.10	7.90	38.10	40.90	1.70	15.30	0.1061	0.6517
0.20	5.20	7.80	38.20	40.80	2.40	14.60	0.1098	0.4790
0.30	5.30	7.70	38.30	40.70	3.10	13.90	0.1138	0.3838
0.40	5.40	7.60	38.40	40.60	3.80	13.20	0.1182	0.3233
0.50	5.50	7.50	38.50	40.50	4.50	12.50	0.1231	0.2814
0.60	5.60	7.40	38.60	40.40	5.20	11.80	0.1285	0.2504
0.70	5.70	7.30	38.70	40.30	5.90	11.10	0.1345	0.2266
0.80	5.80	7.20	38.80	40.20	6.60	10.40	0.1412	0.2077
0.90	5.90	7.10	38.90	40.10	7.30	9.70	0.1488	0.1923
1.00	6.00	7.00	39.00	40.00	8.00	9.00	0.1575	0.1794

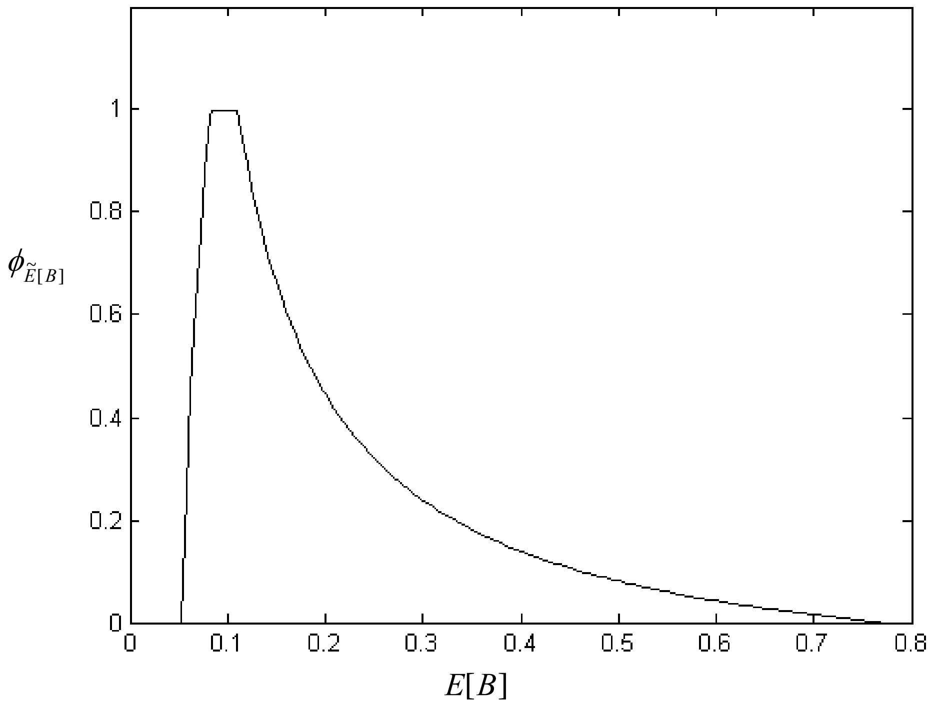

Figure 7. The membership function for fuzzy expected length of time the server is busy

Table 7. α-cuts of arrival, service and setup rates and expected length of time the server is busy

α	x_α^L	x_α^U	y_α^L	y_α^U	v_α^L	v_α^U	$(E[B])_\alpha^L$	$(E\{B\})_\alpha^U$
0.00	5.00	8.00	38.00	41.00	1.00	16.00	0.0524	0.7727
0.10	5.10	7.90	38.10	40.90	1.70	15.30	0.0543	0.4616
0.20	5.20	7.80	38.20	40.80	2.40	14.60	0.0563	0.3319
0.30	5.30	7.70	38.30	40.70	3.10	13.90	0.0586	0.2606
0.40	5.40	7.60	38.40	40.60	3.80	13.20	0.0610	0.2155
0.50	5.50	7.50	38.50	40.50	4.50	12.50	0.0637	0.1844
0.60	5.60	7.40	38.60	40.40	5.20	11.80	0.0668	0.1616
0.70	5.70	7.30	38.70	40.30	5.90	11.10	0.0701	0.1442
0.80	5.80	7.20	38.80	40.20	6.60	10.40	0.0740	0.1304
0.90	5.90	7.10	38.90	40.10	7.30	9.70	0.0783	0.1192
1.00	6.00	7.00	39.00	40.00	8.00	9.00	0.0833	0.1100

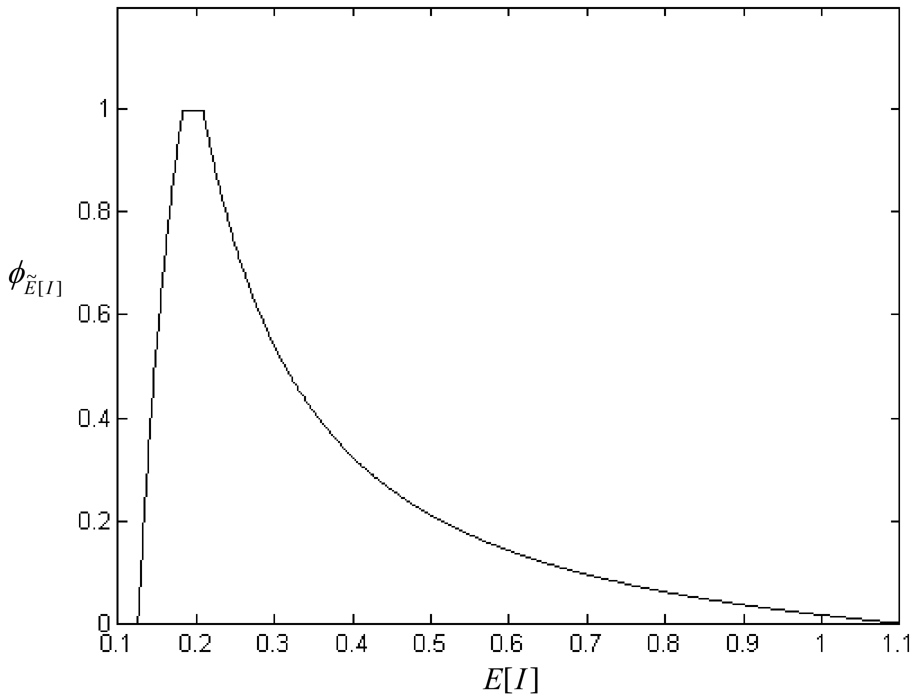

Figure 8. The membership function for fuzzy expected length of time the server is idle

Table 8. α-cuts of arrival and setup rates and expected length of time the server is idle

α	x_α^L	x_α^U	v_α^L	v_α^U	$(E[I])_\alpha^L$	$(E[I])_\alpha^U$
0.00	5.00	8.00	1.00	16.00	0.1250	1.1000
0.10	5.10	7.90	1.70	15.30	0.1287	0.6863
0.20	5.20	7.80	2.40	14.60	0.1326	0.5128
0.30	5.30	7.70	3.10	13.90	0.1369	0.4169
0.40	5.40	7.60	3.80	13.20	0.1415	0.3558
0.50	5.50	7.50	4.50	12.50	0.1467	0.3131
0.60	5.60	7.40	5.20	11.80	0.1523	0.2816
0.70	5.70	7.30	5.90	11.10	0.1586	0.2572
0.80	5.80	7.20	6.60	10.40	0.1656	0.2377
0.90	5.90	7.10	7.30	9.70	0.1735	0.2217
1.00	6.00	7.00	8.00	9.00	0.1825	0.2083

Comparison Analysis Results of Using Fuzzy Theory and Conventional Method

As shown on these two examples, these recorded values of the parameters are approximated to the constants. It is not proper to analyze the characters of the system with a single crisp mean value. Traditionally, engineers used the mean of these possible observations for the parameters as the estimates for finding features such as the expected number of materials in the system, or the expected waiting time in the queue. Example 1 in Section 4, if the traditional approach is used to find the expected number of materials in the system, the arrival rate, service rate and setup rate are respectively estimated as $\hat{\lambda} = 6.5$, $\hat{\mu} = 24.5$, and $\hat{\theta} = 2.5$, and the expected number of materials in the system will be 7.4608. However, if the fuzzy parameters are used to find the expected number of materials in the system, then the ranges are between 3.75 to 20.5714. Table 1 illustrates 10 crisp intervals for fuzzy expected number of materials in the system at different possibilistic α levels. Although the possibilities of some occurrences are very low, the system is still affected by these occurrences. Therefore, the occurrences can not be neglected. If the engineers prefer a suitable single value expected number of materials in the system for practical use with these approximate parameters, this paper uses an approach following the Yager ranking index method. Based on the illustrated example, the suitable expected number of materials in the system is 9.3816. Compared to the expected number of materials in the system with the traditional approach, the risk of under-estimation occurs.

5. Conclusions

This paper applies the concepts of α-cuts and Zadeh's extension principle to a queuing system with a setup and constructs membership functions of the expected number of customers in the system, the expected waiting time in the queue, and the expected lengths of time the server is busy and idle using paired NLP models. Following the proposed approach, α-cuts of the membership functions are found and their interval limits inverted to attain explicit closed-form expressions for the system characteristics. Even when the membership function intervals cannot be inverted, system managers or designers can specify the system characteristics of interest, perform numerical experiments to examine the corresponding α-cuts and then use this information to develop or improve system processes.

For example, in *Example 1*, a designer (manager) can set the range of the expected waiting time to be [0.4199,0.7263] to reflect the desired service and setup rates and find that the corresponding α level is 0.8 with $y_\alpha^L = 23.8$, $y_\alpha^U = 25.2$, $v_\alpha^L = 1.8$ and $v_\alpha^U = 3.2$. In other words, the designer can determine that the service rate is between 23.8 and 25.2 and the setup rate is between 1.8 and 3.2. Similarly, a designer can also set the number of materials with "rounder" numbers like [4.8716, 12.6136] to reflect the desired service and setup rates, and the corresponding α level is 0.6 with $y_\alpha^L = 23.6$, $y_\alpha^U = 25.4$, $v_\alpha^L = 1.6$ and $v_\alpha^U = 3.4$. As this example demonstrates, the approach proposed in this paper provides practical information for system designers and practitioners.

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