Numerical Solution of Fifth Order Boundary Value Problems by Collocation Method with Sixth Order B-Splines

K.N.S. Kasi Viswanadham*, a, P. Murali Krishnaa, C. Prabhakara Raob

a *Department of Mathematics National Institute of Technology, Warangal – 506 004, India*

b *Department of Mathematics, Vasavi College of Engineering, Hyderbad, India*

Abstract: Collocation method with sixth degree B-splines as basis functions has been developed to solve a fifth order special case boundary value problem. To get an accurate solution by the collocation method with sixth degree B-splines, the original sixth degree B-splines are redefined into a new set of basis functions which in number match with the number of collocation points. The method is tested for solving both linear and nonlinear boundary value problems. The proposed method is giving better results when compared with the methods available in literature.

Keywords: Collocation method; sixth degree B-spline; basis function; fifth order boundary value problem; collocation point; absolute error.

1. Introduction

In the present paper we developed a collocation method with sixth degree B-splines as basis functions for getting the numerical solution of fifth order special case boundary value problems, which are in the form

$$y^{(5)}(x) + f(x)y(x) = g(x) \quad a < x < b \quad (1.a)$$

subject to
$$y(a) = a_0, \ y(b) = b_0, \ y'(a) = a_1,$$
$$y'(b) = b_1, \ y''(a) = a_2 \quad (1.b)$$

where a_0, a_1, a_2, b_0, b_1 are finite real constants and $f(x)$, $g(x)$ are continuous functions on [a,b]. Generally, these types of differential equations arise in the mathematical modeling of viscoelastic fluids [1,2]. The existence and uniqueness of the solution for these types of problems has been discussed by Agarwal [3]. Cargal et. al.[4] solved fifth order boundary

value problems by collocation method with sixth degree B-splines. They divided the domain into n subintervals by means of $n+1$ distinct points. They approximated the solution using the given five boundary conditions and the residual is made equal to zero at the inner collocation points only. Ghajala & Shahid [5] solved the boundary value problems of type (1) using sixth degree spline curves. Lamini et. al. [6] developed two methods for the solution of boundary value problems of type (1).

The objective of this paper is to present a simple technique to solve the boundary value problem of the fifth order differential equation (1.a)-(1.b). In the present paper sixth degree B-splines have been used to solve the boundary value problems of the type (1), which are assumed to have a unique solution in the interval of integration [3]. In section 2 of this

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paper the definition of sixth degree B-splines has been described. In section 3, the description of the collocation method with sixth degree B-splines as basis functions has been presented. In section 4, solution procedure to find the nodal parameters has been presented. In section 5, the proposed method is tested on two linear problems and one non-linear problem. The solution of non-linear problem has been obtained as the limit of sequence of linear problems generated by the quasilinearization technique [7]. Finally in the last section the conclusions of the paper are presented.

2. Justification for using collocation method

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Ritz's approach, Galerkin's approach, Least Squares method and Collocation method etc.,. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points (collocation points) in the given space variable domain where N is the number of basis func-

tions in the basis [10]. That means, to get an accurate solution by the collocation method one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain and also the collocation method is the easiest to implement among the variational methods of FEM. The collocation method with cubic B-splines as basis functions has been used to solve a second order boundary value problem [9]. When a differential equation is approximated by m^{th} order B-splines, then it yields (m+1)th order accurate results [9]. It then motivates us to use the collocation method to solve a fifth order boundary value problem of type (1) with sixth order B-splines.

3. Definition of sixth degree b-splines

The cubic B-splines are defined in [8,9]. In a similar analogue, the existence of the sixth degree spline interpolate $s(x)$ to a function in a closed interval $[a, b]$ for spaced knots $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$ is established by constructing it. The construction of s(x) is done with the help of the sixth degree B-splines. Introduce twelve additional knots

$x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and x_{n+6} such that

$x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2}, < x_{-1} < x_0$ and $x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}$.

Now the sixth degree B-splines $B_i(x)$ are defined by

$$B_i(x) = \sum_{r=i-3}^{i+4} \frac{(x_r - x)_+^6}{\pi'(x_r)}, \quad x \in [x_{i-3}, x_{i+4}] \qquad (2.a)$$
$$\qquad = 0 \qquad \qquad other wise$$

where

$$(x_r - x)_+^6 = (x_r - x)^6, \quad if \quad x_r \geq x$$
$$\qquad \qquad = 0 \qquad if \quad x_r \leq x \qquad (2.b)$$

and

$$\pi(x) = \prod_{r=i-3}^{i+4}(x - x_r) \qquad\qquad (2.c)$$

Here the set $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x) \cdots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sixth degree polynomial splines [9]. The sixth degree B-splines are the unique non zero splines of smallest compact support with knots at

$x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}.$

4. Description of the method

To solve the fifth order special case boundary value problem (1) by collocation method with sixth degree B-splines as basis functions, we define the approximation for y(x) as

$$y(x) = \sum_{j=-3}^{n+2}\alpha_j B_j(x) \qquad\qquad (3)$$

where α_j's are the nodal parameters to be determined.

To apply collocation method one has to select the collocation points in the given space variable domain. Here we have taken the mesh points as the selected collocation points. In the approximation (3), we can observe that the number of basis functions is $n+6$. But the number of mesh points in the space variable domain is $n+1$. So there is a necessity to redefine the basis functions into a new set, which should contain $n+1$ basis functions. For this, we proceed in the following manner.

Using the definition of sixth order B-splines described in section 3 and the boundary conditions (1.b), we get the approximation for $y(x)$, $y'(x)$ and $y''(x)$ at the boundary points as

$$y(a) = y(x_0) = \sum_{i=-3}^{2}\alpha_i B_i(x_0) = a_0 \qquad (4)$$

$$y(b) = y(x_n) = \sum_{i=n-3}^{n+2}\alpha_i B_i(x_n) = b_0 \qquad (5)$$

$$y'(a) = y'(x_0) = \sum_{i=-3}^{2}\alpha_i B_i'(x_0) = a_1 \qquad (6)$$

$$y'(b) = y'(x_n) = \sum_{i=n-3}^{n+2}\alpha_i B_i'(x_n) = b_1 \qquad (7)$$

$$y''(a) = y''(x_0) = \sum_{i=-3}^{2}\alpha_i B_i''(x_0) = a_2 \qquad (8)$$

Eliminating $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_{n+1}$ and α_{n+2} from the equations (3) to (8), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=0}^{n}\alpha_j \tilde{B}_j(x) \qquad (9)$$

where

$$w(x) = w_2(x) + \frac{Q_{-1}(x)}{Q_{-1}''(x_0)}\left(a_2 - w_2''(x_0)\right) \qquad (10)$$

$$w_2(x) = w_1(x) + \frac{\left(a_1 - w_1'(x_0)\right)}{P_{-2}'(x_0)}P_{-2}(x) + \frac{\left(b_1 - w_1'(x_n)\right)}{P_{n+1}'(x_n)}P_{n+1}(x) \qquad\qquad (11)$$

$$w_1(x) = \frac{B_{-3}(x)}{B_{-3}(x_0)}a_0 + \frac{B_{n+2}(x)}{B_{n+2}(x_n)}b_0 \qquad\qquad (12)$$

$$\tilde{B}_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_{-1}(x)}{Q_{-1}''(x_0)}Q_j''(x_0) & \text{if } j \leq 2 \\[2mm] Q_i(x) & \text{if } j \geq 3 \end{cases} \qquad\qquad (13)$$

$$Q_i(x) = \begin{cases} P_j(x) - \dfrac{P_{-2}(x)}{P'_{-2}(x_0)} P'_j(x_0) & \text{if } j \leq 2 \\ P_j(x) & \text{if } j = 3,4,...,n-2 \\ P_j(x) - \dfrac{P_{n+1}(x)}{P'_{n+1}(x_n)} P'_j(x_n) & \text{if } j \geq n-3 \end{cases} \qquad (14)$$

$$P_i(x) = \begin{cases} B_j(x) - \dfrac{B_{-3}(x)}{B_{-3}(x_0)} B_j(x_0) & \text{if } j \leq 2 \\ B_j(x) & \text{if } j = 3,4,...,n-2 \\ B_j(x) - \dfrac{B_{n+2}(x)}{B_{n+2}(x_n)} B_j(x_n) & \text{if } j \geq n-3 \end{cases} \qquad (15)$$

Now the new set of basis functions is {$\tilde{B}_j(x)$, $j = 0,1,2,…,n$} and the number of basis functions in number match with the number of selected collocation points..

Applying the collocation method with the redefined set of basis functions $\tilde{B}_j(x)$, $j = 0,1,2,…,n$ to the problem (1), we get

$$\left\{ \left. \dfrac{d^5 w}{dx^5} \right|_{x_i} + \sum_{j=0}^{n} \alpha_j \left. \dfrac{d^5 \tilde{B}_j}{dx^5} \right|_{x_i} \right\} + f(x_i) \left\{ w(x_i) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j(x_i) \right\} = g(x_i), \quad i = 0,1,2,…,n \qquad (16)$$

Rewriting the above system of equations in the matrix form, we get $\quad A\alpha = b \qquad (17)$
where

$$\mathbf{A} = [\,a_{ij}\,]; \quad a_{ij} = \left. \dfrac{d^5 \tilde{B}_j}{dx^5} \right|_{x_i} + f(x_i)\tilde{B}_j(x_i) \quad \text{, for } i = 0,1,2,…,n \quad \text{and } j = 0,1,2,…,n \qquad (18)$$

$$\mathbf{b} = [\,b_i\,]; \quad b_i = g(x_i) - \left. \dfrac{d^5 w}{dx^5} \right|_{x_i} - f(x_i)w(x_i), \quad \text{for } i = 0,1,2,…,n \qquad (19)$$

$$\alpha = \begin{bmatrix} \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n \end{bmatrix}^T \qquad (20)$$

4. Solution procedure to find the nodal parameters

The basis function $\tilde{B}_j(x)$ is defined only in the interval [x_{j-3}, x_{j+4}] and outside of this interval it is zero. Also at the end points of the interval [x_{j-3}, x_{j+4}] the basis function $\tilde{B}_j(x)$ vanishes. Therefore, $\tilde{B}_j(x)$ is having non–vanishing values at the mesh points x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, x_{j+3} and at other mesh points the value of $\tilde{B}_j(x)$ is zero. The derivatives of $\tilde{B}_j(x)$ upto fifth order also have the same nature at the mesh points as in the case of $\tilde{B}_j(x)$. Using these facts, we can say that the matrix **A** defined in (18) is a six band matrix. Therefore, the system of equations (17) is a six band system in α_j's. The nodal parameters α_j's can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problems (1) by the proposed method.

5. Numerical Examples

To demonstrate the applicability of the proposed method for solving the fifth order special case boundary value problems of type (1), we considered three examples of two linear boundary value problems and one non linear boundary value problem. These examples have been chosen because either analytical or approximate solutions are available in the literature and solutions obtained by the proposed method are compared with the solutions obtained by the methods available in the literature.

Example 1: Consider the following fifth order linear boundary value problem

$y^{(5)}(x) - y(x) = -(15+10x)\, e^x, 0 \le x \le 1$ (21)
subject to
$y(0) = y(1)= 0, y'(0)=1, y'(1) = - e, y^{(2)}(0) = 0.$

The exact solution for the above system is given by $y(x) = x(1 - x)\, e^x$.

We have solved the problem (21) by the proposed method. We have taken the number of intervals for the space variable domain as 10. The approximate solution obtained by the proposed method is compared with the exact solution in Table 1. The maximum absolute error obtained by the proposed method is compared with that of obtained by Caglar et. al.[4], Ghajala & Shahid [5] in Table 2.

Table 1. Numerical results for Example-1

x	Solution by the proposed method	Exact solution
0.1	0.0994654	0.0994654
0.2	0.1954271	0.1954244
0.3	0.2834731	0.2834703
0.4	0.3580460	0.3580379
0.5	0.4121955	0.4121803
0.6	0.4373200	0.4373085
0.7	0.4228971	0.4228881
0.8	0.3560894	0.3560865
0.9	0.2213583	0.2213643

Table 2. Maximum Absolute Error (MAE) $| y(x_i) - y_i |$ for Example 1

Caglar et al [4] with $h = 1/10$	Ghajala & Shahid [5] with $h = 1/10$	Proposed method with $h = 1/10$
0.01570	2.2593×10^{-4}	1.5228×10^{-5}

Example 2: Consider the following fifth order linear boundary value problem
$y^{(5)}(x) + xy(x) = 19\, x \cos(x) + 2x^3 \cos(x) + 41 \sin(x) - 2x^2 \sin(x),$ $-1 \le x \le 1$ subject to $y(-1) = y(1) = \cos(1),$ $y'(-1) = -y'(1) = -4\cos(1) + \sin(1), y^{(2)}(-1) = 3 \cos(1) - 8 \sin(1).$ (22)

The exact solution for the above system is given by $y(x) = (2x^2 - 1) \cos(x).$
We have solved the problem (22) by the proposed method. We have taken the number of in-

tervals for the space variable domain as 10. The approximate solution obtained by the proposed method is compared with the exact solution in Table 3. The maximum absolute error obtained by the proposed method is 4.5162×10^{-4}.

Table 3. Numerical results for Example-2

x	Solution by the proposed method	Exact solution
-0.8	0.1950971	0.1950778
-0.6	-0.2309726	-0.2310939
-0.4	-0.6260466	-0.6263214
-0.2	-0.9012573	-0.9016612
0.0	-0.9995484	-1.0000000
0.2	-0.9012666	-0.9016612
0.4	-0.6260669	-0.6263214
0.6	-0.2309932	-0.2310939
0.8	0.1950898	0.1950778

Example 3: Consider the following fifth order nonlinear boundary value problem

$$y^{(5)}(x) + 24\, e^{-5y} = \frac{48}{(1+x)^5}, \qquad 0 \le x \le 1 \tag{23}$$

subject to $y(0) = 0,\ y(1) = \ln(2), \quad y'(0) = 1,\ y'(1) = 0.5, \quad y^{(2)}(0) = -1 \tag{24}$

The exact solution for the above system is given by $y(x) = \ln(1 + x)$.

Applying the quasilinearization technique [7] to the equation (23), we get a sequence of linear problems as

$$y_{r+1}^{(5)}(x) - 120\exp(-5y_r)\,y_{r+1} = \frac{48}{(1+x)^5} - 120 y_r \exp(-5y_r) - 24\exp(-5y_r),\ \text{for}\quad r = 0,1,2, \tag{25}$$

We have solved the sequence of problems (25) along with boundary conditions (24) by the proposed method. We have taken the number of intervals for the space variable domain as 10. The approximation to the solution is converged in two iterations. The approximate solution obtained by the proposed method is compared with the exact solution in Table 4. The maximum absolute error obtained by the proposed method is compared with that of obtained by Caglar et. al.[4] in Table 5.

6.   Conclusions

In this paper, we have developed a colloca-tion method with sixth order B-splines as basis functions to solve fifth order special case boundary value problems. In the collocation method, we have selected the mesh points as collocation points. The sixth order B-spline basis set has been redefined into a new set in which the number of basis functions is equal to the number of collocation points. The proposed method is applied to solve two linear problems and one non-linear problem to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The maximum absolute errors obtained by the proposed method are less when compared with those of avail-

able in the literature. The objective of this paper is to present a simple and direct technique to solve a fifth order special case boundary value problem. The proposed method can be extended to solve higher (i.e., more than 5th) order boundary value problems by using the B-splines of order more than 6.

Table 4. Numerical results for Example-3

X	Solution by the proposed method	Exact solution
0.1	0.0953113	0.0953102
0.2	0.1823308	0.1823216
0.3	0.2623831	0.2623643
0.4	0.3365076	0.3364722
0.5	0.4055183	0.4054651
0.6	0.4700552	0.4700036
0.7	0.5306774	0.5306283
0.8	0.5878178	0.5877867
0.9	0.6418715	0.6418539

Table 5. Maximum Absolute Error (MAE) $| y(x_i) - y_i |$ for Example 3

Caglar et al [4] with $h = 1/30$	Proposed method with $h = 1/10$
0.046	5.3197×10^{-5}

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