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Two Phase Method for Solving Fuzzy Linear Programming Problems using Ranking of Generalized Fuzzy Numbers

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Abstract: Ranking of fuzzy numbers play an important role in decision making problems. Fuzzy numbers must be ranked before an action is taken by a decision maker. Chen (Operations on fuzzy numbers with function principal, Tamkang Journal of Management Science 6 (1985) 13-25) pointed out that in many cases it is not to possible to restrict the membership function to the normal form and proposed the concept of generalized fuzzy numbers. In this paper two phase method is proposed for solving a special type of fuzzy linear programming (FLP) problems using generalized fuzzy numbers. To illustrate the proposed method a numerical example is solved and the advantages of the proposed method are discussed. Since the proposed method is a direct extension of classical method so it is very easy to understand and apply the proposed method to find the fuzzy optimal solution of FLP problems occurring in the real life situations.

Keywords: fuzzy linear programming problems; ranking function; trapezoidal fuzzy numbers.

1. Introduction

Linear programming is one of the most frequently applied operations research techniques. Although it is investigated and expanded for more than six decades by many researchers and from the various point of views, it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming. Any linear programming model representing real world situations involves a lot of parameters whose values are assigned by experts, and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However, both experts and the decision makers frequently do not precisely know the value of those parameters. If exact values are suggested these are only statistical inference from past data and their stability is doubtful, so the parameters of the problem are usually defined by the decision makers in an uncertain way or by means of language statement parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data [50].

The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. [45] in the frame work of the fuzzy decision of Bellman and Zadeh [2]. The first formulation of FLP was proposed by Zimmermann [51]. Afterwards, many authors [3, 5, 17, 21, 23, 24, 26, 32, 36, 42-44] considered various types of the FLP problems and proposed several approaches for solving these problems.

Fuzzy numbers must be ranked before an

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action is taken by a decision maker. Real numbers can be linearly ordered by the relation $\leq$ or $\geq$, however this type of inequality does not exist in fuzzy numbers. Since fuzzy numbers are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than other. An efficient method for ordering the fuzzy numbers is by the use of a ranking function, which maps each fuzzy number into the real line, where a natural order exists. Jain [25] proposed the concept of ranking function for comparing normal fuzzy numbers. Chen [7] pointed out that in many cases it is not to possible to restrict the membership function to the normal form and proposed the concept of generalized fuzzy numbers. Since then, tremendous efforts are spent; significant advances are made on the development of numerous methodologies [4-6, 8-10, 12-16, 20, 22, 30, 31, 46, 47, 48] for the ranking of generalized fuzzy numbers. Chen and Chen [10] pointed out the shortcomings of the existing methods for the ranking of generalized fuzzy numbers and proposed a new method. Ranking function is used in different areas of fuzzy optimization [1, 11, 18, 21, 33-35, 37-41, 49].

In this paper two phase method is proposed for solving a special type of FLP problem. To illustrate the proposed method a numerical example is solved and the advantages of the proposed method are discussed. Since the proposed method is a direct extension of classical method so it is very easy to understand and apply the proposed method to find the fuzzy optimal solution of FLP problems occurring in the real life situations.

This paper is organized as follow: In Section 2, importance of generalized fuzzy numbers are discussed. In Section 3, some basic definitions and arithmetic operations between generalized trapezoidal fuzzy numbers are reviewed. In Section 4, a method for comparing two generalized trapezoidal fuzzy numbers is presented. In Section 5, formulation and definitions of FLP problems are discussed. In Section 6, two phase method is proposed for solving a special type of FLP problems. To explain the proposed method a numerical example is solved in Section 7. In Section 8, advantages of the proposed method are presented. In the last section conclusion is discussed.

2. Why generalized fuzzy numbers?

In most of the papers generalized fuzzy numbers are converted into normal fuzzy numbers through normalization process [27] and then obtained normal fuzzy numbers are used to solve the real life problems. Kaufmann and Gupta [27] pointed out that there is a serious disadvantage of the normalization process. Basically we have transformed a measurement of an objective value to a valuation of a subjective value, which results in the loss of information. Although this procedure is mathematically correct but it decreases the amount of information that is available in the original data, and we should avoid it.

Hsieh and Chen [22] pointed out that arithmetic operators on fuzzy numbers, presented in Chen [7], does not only change the type of membership function of fuzzy numbers, but they can also reduce the troublesomeness and tediousness of arithmetic operations. There are several papers [8-10, 14, 34, 48] in which generalized fuzzy numbers are used for solving real life problems.

3. Preliminaries

In this section some basic definitions and arithmetic operations are reviewed [19,27].

3.1. Basic definitions

Definition 3.1 A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be generalized fuzzy number if its membership function has the following characteristics:

(i) $\mu_{\tilde{A}} : R \to [0,1]$ is continuous.

(ii) $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a] \cup [d, \infty)$.

(iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$ and strictly decreasing on $[c,d]$.

(iv) $\mu_{\tilde{A}}(x) = w,$ for all $x \in [b,c],$ where $0 < w \le 1$.

Definition 3.2 A generalized fuzzy number $\tilde{A} = (a,b,c,d;w)$ is said to be a generalized trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{w(x-a)}{(b-a)} & ,a \le x \le b \\ w & ,b \le x \le c \\ \dfrac{w(x-d)}{(c-d)} & ,c \le x \le d \end{cases}$$

Remark 3.1. If the data is collected from an expert and on the basis of collected data the cost (or demand or supply or profit etc.) of the product is represented by a generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ then it can be explained as follows:

(i) According to decision maker the cost (or demand or supply or profit etc.) of the product will be greater than a units and less than d units.

(ii) Decision maker is $w \times 100\%$ in favour that the cost (or demand or supply or profit etc.) will be greater than or equal to b units and less than or equal to c units.

(iii) The percentage of the favourness of the decision maker for the remaining values of cost (or demand or supply or profit etc.) can be obtained as follows:

Let x represents the cost (or demand or supply or profit etc.) then the percentage of the favourness of the decision maker for $x = \mu_{\tilde{A}}(x) \times 100$,

where $$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{w(x-a)}{(b-a)} & ,a \le x \le b \\ w & ,b \le x \le c \\ \dfrac{w(x-d)}{(c-d)} & ,c \le x \le d \end{cases}$$

3.2. Arithmetic operations

In this section, arithmetic operations between two generalized trapezoidal fuzzy numbers, defined on universal set of real numbers R, are reviewed [10].

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2))$

(ii) $\tilde{A}_1 \Theta \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2))$

(iii) $\tilde{A}_1 \otimes \tilde{A}_2 = (a', b', c', d'; \min(w_1, w_2)),$ where $a' = \min(a_1a_2, a_1d_2, a_2d_1, d_1d_2)$
$b' = \min(b_1b_2, b_1c_2, c_1b_2, c_1c_2),$ $c' = \max(b_1b_2, b_1c_2, c_1b_2, c_1c_2)$ $d' = \max(a_1a_2, a_1d_2, a_2d_1, d_1d_2)$

(v) $\lambda \tilde{A}_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1; w_1), \lambda \geq 0 \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1; w_1), \lambda < 0 \end{cases}$

4. Ranking function

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function $\Re : F(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists.

Several authors have proposed different methods for the ranking of generalized fuzzy numbers but it can be easily seen from the literature that whenever some one have proposed a new method for the ranking of generalized fuzzy numbers, the other authors or some times the same author have pointed out the shortcomings of the proposed methods. For example Chen and Chen [9] pointed out the shortcomings of several existing methods and proposed a new method. In 2009, Chen and Chen [10] again proposed a method in which they pointed out the shortcomings of their method [9]. Kumar et al. [30] have pointed the shortcomings of Chen and Chen [10] method but to the best of our knowledge till now no one have pointed out any shortcoming in the Liou and Wang [31] ranking approach. Also, it is easy to apply the Liou and Wang [31] ranking formula as compared to the other existing ranking formula.

Since value of rank is calculated from the extreme values of $\lambda -$cut of $\tilde{A}$, rather than its membership function, it is not required knowing the explicit form of the membership functions of the fuzzy numbers to be ranked. That is, unlike most of the ranking methods that require the knowledge of the membership functions of all fuzzy numbers to be ranked, the Liou and Wang [31] ranking method is still applicable even if the explicit form of membership function of the fuzzy number is unknown. Due to the above described reasons in this paper the existing ranking formula [31] is used.

4.1. Method to compare two generalized trapezoidal fuzzy numbers

In this section, the method, used in the numerical examples, to compare two generalized trapezoidal fuzzy numbers [31] is presented.

In this paper to compare two generalized trapezoidal fuzzy numbers, the following method is used.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers then use the following Steps to compare $\tilde{A}$ and $\tilde{B}$.

Step 1 Find $w = \min(w_1, w_2)$

Step 2 Find $\Re(\tilde{A}) = w\left(\frac{a_1 + b_1 + c_1 + d_1}{4}\right)$ and $\Re(\tilde{B}) = w\left(\frac{a_2 + b_2 + c_2 + d_2}{4}\right)$

Case (i) If $\Re(\tilde{A}) > \Re(\tilde{B})$ then $\tilde{A} \underset{\Re}{>} \tilde{B}$ i.e., minimum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$.

Case (ii) If $\Re(\tilde{A}) < \Re(\tilde{B})$ then $\tilde{A} \underset{\Re}{<} \tilde{B}$ i.e., minimum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$.

Case (iii) If $\Re(\tilde{A}) = \Re(\tilde{B})$ then $\tilde{A} \underset{\Re}{\approx} \tilde{B}$ i.e., minimum $\{\tilde{A}, \tilde{B}\} = \tilde{A} = \tilde{B}$.

Example 4.1 Let $\tilde{A}=(-6,-2,3,11;0.2)$, $\tilde{B}=(-7,4,10,16;0.4)$ and $\tilde{C}=(4,7,8,10;0.5)$
Now compare $\tilde{A}\oplus\tilde{B}$ and $\tilde{C}$ we will use the following Steps:

Step 1 Find $w=\min(0.2,0.5)=0.2$ and calculate $\tilde{A}\oplus\tilde{B}=(-13,2,13,27;0.2)$

Step 2 Now $\Re(\tilde{A}\oplus\tilde{B})=1.45$ and $\tilde{C}=1.45$. Since $\Re(\tilde{A}\oplus\tilde{B})=\Re(\tilde{C})$ So, $\tilde{A}\oplus\tilde{B}\approx\tilde{C}$.

Similarly, if we compare $\tilde{A}\oplus\tilde{C}=(-2,5,11,21;0.2)$ and $\tilde{B}=(-7,4,10,16;0.2)$ then $\Re(\tilde{A}\oplus\tilde{C})=1.75$ and $\Re(\tilde{B})=1.15$. So, $\tilde{A}\oplus\tilde{C}\underset{\Re}{>}\tilde{B}$.

Note: $\tilde{0}$ represents a generalized trapezoidal fuzzy number whose rank is zero.

5. Fuzzy linear programming problem

Linear programming is one of the most frequently applied operations research technique. In the conventional approach value of the parameters of linear programming models must be well defined and precise. However, in real world environment, this is not a realistic assumption.

In the real life problems the following situation may occur:
If a company want to launch new product in the market then there may exist uncertainty about the profit (or cost) and availability (or demand) of the product. In such a situation the profit (or cost) and availability (or demand) may be represented by generalized fuzzy numbers. In such a case the real life problems, of m fuzzy constraints and n fuzzy variables, may be formulated as follow:

$$Maximize\ (or\ Minimize)\tilde{z}\underset{\Re}{\approx}\tilde{C}^T\otimes\tilde{X}$$

$$\text{subject to}\quad A\tilde{X}\underset{\Re}{\leq},\underset{\Re}{\approx},\underset{\Re}{\geq},\tilde{b}$$

$$\tilde{X}\underset{\Re}{\geq}0 \qquad (1)$$

where
$\tilde{C}^T=[\tilde{c}_j]_{1\times n},\tilde{X}=[\tilde{x}_j]_{n\times 1},A=[a_{ij}]_{m\times n},\tilde{b}=[\tilde{b}_i]_{m\times 1}$
and $a_{ij}\in R,\tilde{c}_j,\tilde{x}_j,\tilde{b}_i\in F(R)$.

Definition 5.1. [33] Let the i^{th} fuzzy constraint of a FLP problem be $\sum_{j=1}^{n}a_{ij}\tilde{x}_j\underset{\Re}{\leq}\tilde{b}_i$ where $\tilde{b}_i\underset{\Re}{\geq}0$ then a fuzzy variable $\tilde{s}_i$ such that $\tilde{s}_i\underset{\Re}{\geq}0$ and $\sum_{j=1}^{n}a_{ij}\tilde{x}_j\oplus\tilde{s}_i\underset{\Re}{\approx}\tilde{b}_i$ is called a fuzzy slack variable.

Definition 5.2. [33] Let the i^{th} fuzzy constraint of a FLP problem be $\sum_{j=1}^{n}a_{ij}\tilde{x}_j\underset{\Re}{\geq}\tilde{b}_i$ where $\tilde{b}_i\underset{\Re}{\geq}0$ then a fuzzy variable $\tilde{s}_i$ such that $\tilde{s}_i\underset{\Re}{\geq}0$ and $\sum_{j=1}^{n}a_{ij}\tilde{x}_j\Theta\tilde{s}_i\underset{\Re}{\approx}\tilde{b}_i$ is called a fuzzy surplus variable.

Definition 5.3. [23] Given a system of m fuzzy linear equation involving generalized trapezoidal fuzzy numbers in n unknowns $(m\leq n), A\tilde{X}\underset{\Re}{\approx}\tilde{b}$, where A is a $m\times n$ real matrix and rank of A is m. Let the columns of A corresponding to fuzzy variables $\tilde{x}_{k_1},\tilde{x}_{k_2},...,\tilde{x}_{k_m}$ are linearly independent then $\tilde{x}_{k_1},\tilde{x}_{k_2},...,\tilde{x}_{k_m}$ are said to be fuzzy basic variables and remaining ($n-m$) variables are called fuzzy nonbasic variables. Let B the basis matrix formed by linearly independent columns of A. The value of $\tilde{X}_B=(\tilde{x}_{k_1},\tilde{x}_{k_2},...,\tilde{x}_{k_m})^T$ is obtained by using

$\tilde{X}_B = B^{-1}\tilde{b},$ where $k_j \in \{1,2,...,n\}, k_i \neq k_j, i, j = 1,2,...,m$ and the values of non basic variables are assumed to be zero. The combined solution formed by using the values of fuzzy basic variables and fuzzy nonbasic variables are called fuzzy basic solution.

Definition 5.4. [23] Any $\tilde{X}$ which satisfies all the constraints $(A\tilde{X} \underset{\Re}{\leq}, \underset{\Re}{\approx}, \underset{\Re}{\geq}, \tilde{b})$ and non-negative restrictions ($\tilde{X} \underset{\Re}{\geq} 0$) of (1) is said to be a fuzzy feasible solution of (1).

Definition 5.5. [23] Let $\tilde{S}_F$ be the set of all fuzzy feasible solution of (1). A fuzzy feasible solution $\tilde{X}_0 \in \tilde{S}_F$ is said to be a fuzzy optimal solution of (1) if $\tilde{C}^T \otimes \tilde{X}_0 \underset{\Re}{\geq} \tilde{C}^T \otimes \tilde{X}$ (or $\tilde{C}^T \otimes \tilde{X}_0 \underset{\Re}{\leq} \tilde{C}^T \otimes \tilde{X}$), for all $\tilde{X} \in \tilde{S}_F$

6. Two-phase method

There are several papers in the literature [8-10, 14, 34, 48] in which generalized fuzzy numbers are used for solving real life problems but to the best of our knowledge, till now no one has used generalized fuzzy numbers for solving the linear programming problems with fuzzy parameters.

6.1. Why two phase method?

Nasseri and Ardil [38] proposed a fuzzy simplex method for solving the following type of FLP problems:

$$Maximize\ (or\ Minimize)\tilde{z} \underset{\Re}{\approx} \tilde{C}^T \otimes \tilde{X}$$

$$\text{subject to } A\tilde{X} \underset{\Re}{\leq} \tilde{b}$$

$$\tilde{X} \underset{\Re}{\geq} 0$$

where

$\tilde{C}^T = [\tilde{c}_j]_{1\times n}, \tilde{X} = [\tilde{x}_j]_{n\times 1}, A = [a_{ij}]_{m\times n}, \tilde{b} = [\tilde{b}_i]_{m\times 1}$ and $a_{ij} \in R, \tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in F(R).$

The existing method [38] can't be applied for nd the optimal solution of the following type of FLP problems

$$Maximize\ (or\ Minimize)\tilde{z} \underset{\Re}{\approx} \tilde{C}^T \otimes \tilde{X}$$

$$\text{subject to } A\tilde{X} \underset{\Re}{\geq} \tilde{b}, \quad \tilde{X} \underset{\Re}{\geq} 0$$

where

$\tilde{C}^T = [\tilde{c}_j]_{1\times n}, \tilde{X} = [\tilde{x}_j]_{n\times 1}, A = [a_{ij}]_{m\times n}, \tilde{b} = [\tilde{b}_i]_{m\times 1}$ and $a_{ij} \in R, \tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in F(R).$

To find the fuzzy optimal solution of above type of FLP problems a fuzzy two phase method is introduced.

6.2. Proposed method

In this section two phase method is proposed to obtain the fuzzy optimal solution for the following type of FLP problem:

$$Maximize\ (or\ Minimize)\tilde{z} \underset{\Re}{\approx} \tilde{C}^T \otimes \tilde{X}$$

$$\text{subject to } A\tilde{X} \underset{\Re}{\geq} \tilde{b}, \quad \tilde{X} \underset{\Re}{\geq} 0, \text{ where}$$

$\tilde{C}^T = [\tilde{c}_j]_{1\times n}, \tilde{X} = [\tilde{x}_j]_{n\times 1}, A = [a_{ij}]_{m\times n}, \tilde{b} = [\tilde{b}_i]_{m\times 1}$ and $a_{ij} \in R, \tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in F(R).$

The steps of the proposed algorithm are as follows:

Step 1

Check that $\tilde{x}_j \underset{\Re}{\geq} 0$ or not $\forall\ j = 1,2,...,n.$

Case (i) If $\tilde{x}_j \underset{\Re}{\geq} 0\ \forall\ j$, then Go to Step 2.

Case (ii) If there exist any $\tilde{x}_j$ such that $\tilde{x}_j \underset{\Re}{<} 0$ then replace $\tilde{x}_j$ by $\Theta\tilde{y}_j$ in the given problem so that $\tilde{y}_j \underset{\Re}{>} 0$, then Go to Step 2.

Case (iii) If $\tilde{x}_j \underset{\Re}{\geq} a$, then replace $\tilde{x}_j$ by $\tilde{y}_j \oplus \tilde{a}$, where $\tilde{a}$ is a fuzzy number corresponding to crisp number a, so that $\tilde{y}_j \underset{\Re}{\geq} 0$, then Go to Step 2.
Case (iv) If $\tilde{x}_j \underset{\Re}{\leq} a$, then replace $\tilde{x}_j$ by $\tilde{a} \Theta \tilde{y}_j$, where $\tilde{a}$ is a fuzzy number corresponding to crisp number a, so that $\tilde{y}_j \underset{\Re}{\geq} 0$, then Go to Step 2.
Case (v) If there exist any $\tilde{x}_j$ which is unrestricted, i.e., there is no restriction on the sign of $\Re(\tilde{x}_j)$, then replace $\tilde{x}_j$ by $\tilde{x}'_j \Theta \tilde{x}''_j$ where $\tilde{x}'_j \underset{\Re}{\geq} 0$ and $\tilde{x}''_j \underset{\Re}{\geq} 0$ in the given problem then Go to Step 2.
Case (vi) If $a \underset{\Re}{\leq} \tilde{x}_j \underset{\Re}{\leq} b$, i.e., $a \underset{\Re}{\leq} \tilde{x}_j$ and $\tilde{x}_j \underset{\Re}{\leq} b$, then replace $\tilde{x}_j$ by $\tilde{y}_j \oplus \tilde{a}$ so that $\tilde{y}_j \underset{\Re}{\geq} 0$, then Go Step 2.

Step 2

Check whether the given problem is of maximization or minimization.

Case (i) If it is of maximization then Go to Step 3.

Case (ii) If it is of minimization then convert it into maximization problem by multiplying -1 in the objective function then Go to Step 3.

Step 3

Check that $\tilde{b}_i \underset{\Re}{\geq} 0$ or not $\forall\ i = 1, 2, ..., m.$

Case (i) If $\tilde{b}_i \underset{\Re}{\geq} 0\ \forall\ i$ then Go to Step 4.

Case (ii) If there exist any $\tilde{b}_i$ such that $\tilde{b}_i \underset{\Re}{<} 0$ then multiply corresponding constraint of $\tilde{b}_i$ by -1 so that $\tilde{b}_i \underset{\Re}{>} 0\ \forall\ i$.

Step 4

Convert all the inequalities of the constraints into equation by introducing fuzzy slack and fuzzy surplus variables. Put the coefficients of these fuzzy slack and fuzzy surplus variables equal to zero in the objective function.

Step 5

Let after introducing p fuzzy slack and fuzzy surplus variables the constraints $A\tilde{X} \underset{\Re}{\leq}, \underset{\Re}{\approx}, \underset{\Re}{\geq}, \tilde{b}$, where $A = [a_{ij}]_{m \times n}, \tilde{X} = [\tilde{x}_j]_{n \times 1}, \tilde{b} = [\tilde{b}_i]_{m \times 1}$ and $a_{ij} \in \Re$, $\tilde{x}_j, \tilde{b}_i \in F(R)$, $\tilde{x}_j \underset{\Re}{\geq} \tilde{0}, \tilde{b}_i \underset{\Re}{\geq} \tilde{0}$ be converted into $A'\tilde{X}' \underset{\Re}{\approx} \tilde{b}$, where $A' = [a_{ij}]_{m \times (n+p)}, \tilde{X}' = [\tilde{x}_j]_{(n+p) \times 1}$, $\tilde{b} = [\tilde{b}_i]_{m \times 1}$ and $a_{ij} \in \Re$, $\tilde{x}_j, \tilde{b}_i \in F(R)$, $\tilde{x}_j \underset{\Re}{\geq} \tilde{0}, \tilde{b}_i \underset{\Re}{\geq} \tilde{0}$. Now check whether an identity submatrix of order $m \times m$ exist or not in the coefficient matrix A'.

Case (i) If there exist an identity submatrix A' then fuzzy optimal solution can be obtained by solving $Maximize\ \tilde{z} \underset{\Re}{\approx} \tilde{C}'^T \otimes \tilde{X}'$ $subject\ to\ A'\tilde{X}' \underset{\Re}{\approx} \tilde{b}$, $\tilde{X}' \underset{\Re}{\geq} \tilde{0}, \tilde{b} \underset{\Re}{\geq} \tilde{0}$ using Step 6, Step 7(a) and Step 8, where $(\tilde{C}')^T = [\tilde{c}_j]_{1 \times (n+p)}$ and

$$\tilde{c}' = \begin{cases} \tilde{c}_j, & j = 1, 2, ...n \\ 0, & j = n+1, n+2, ..., n+p \end{cases}$$

Case (ii) If the identity submatrix does not exist then check is it possible to construct an identity submatrix without adding any extra variable and with maintaining the condition $\tilde{b}_i \underset{\Re}{\geq} \tilde{0}\ \forall\ i$.
Case (a) If possible then construct an identity submatrix and obtain the fuzzy optimal solution as in described Case (i).
Case (b) If not then check which identity

column does not exist in the coefficient matrix A' i.e. $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}\cdots$ or $\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$. Let identity column $\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}$, in which 1 occurs at the p^{th} position, does not exist, then add a fuzzy artificial variable x_0 in the p^{th} constraint. Let after adding l fuzzy artificial variables $A'\tilde{X}'\underset{\Re}{\approx}\tilde{b}$, $\tilde{X}'\underset{\Re}{\geq}\tilde{0},\tilde{b}\underset{\Re}{\geq}\tilde{0}$ be converted into $A''\tilde{X}''\underset{\Re}{\approx}\tilde{b}$, $\tilde{X}''\underset{\Re}{\geq}\tilde{0},\tilde{b}\underset{\Re}{\geq}\tilde{0}$, where $A''=[a_{ij}]_{m\times(n+p+l)},\tilde{X}''=[\tilde{x}_j]_{(n+p+l)\times 1}$, $\tilde{b}=[\tilde{b}_i]_{m\times 1}$ and $a_{ij}\in\Re$, $\tilde{x}_j,\tilde{b}_i\in F(R)$, $\tilde{x}_j\underset{\Re}{\geq}\tilde{0},\tilde{b}_i\underset{\Re}{\geq}\tilde{0}$.

Step 6

The fuzzy variables, $\tilde{x}_{B1},\tilde{x}_{B2},...,\tilde{x}_{Bm}$, constituting the identity sub-matrix give the basis $\tilde{X}_B=(\tilde{x}_{B1},\tilde{x}_{B2},...,\tilde{x}_{Bm})$ i.e. columns of $\tilde{x}_{B1},\tilde{x}_{B2},...,\tilde{x}_{Bm}$ in the coefficient matrix A are $(1,0,0,...,0)^T,(0,1,0,...,0)^T,...,(0,0,0,...,1)^T$ respectively. The values of $\tilde{x}_{B1},\tilde{x}_{B2},...,\tilde{x}_{Bm}$ can be obtained by putting the values of remaining $(n+l+p-m)$ fuzzy nonbasic variables equal to zero in $A''\tilde{X}''\underset{\Re}{\approx}\tilde{b}$.

Step 7

Now apply PHASE-I to find the fuzzy basic feasible solution for the system of constraints with the help of an auxiliary objective function.

PHASE-I

Solve the FLP problem

$$Maximize\ \tilde{z}\underset{\Re}{\approx}\Theta\tilde{x}_{(n+p+1)}\Theta\tilde{x}_{(n+p+2)}\Theta...\Theta\tilde{x}_{(n+p+l)}$$

subject to $A''\tilde{X}''\underset{\Re}{\geq}\tilde{b}$

$\tilde{X}''\underset{\Re}{\geq}0,\tilde{b}\underset{\Re}{\geq}0$ where

$A''=[a_{ij}]_{m\times(n+p+l)},\tilde{X}''=[\tilde{x}_j]_{(n+p+l)\times 1}$,

$\tilde{b}=[\tilde{b}_i]_{m\times 1}$.

Step 7 (a)

Construct the fuzzy simplex table in the following format:

Table1. Fuzzy simplex table

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\cdots$	$\tilde{x}_r$	$\cdots$	$\tilde{x}_{(n+p+l)}$
$\tilde{x}_{B1}$	a_{11}	a_{12}	$\cdots$	a_{1r}	$\cdots$	$a_{1(n+p+l)}$
$\tilde{x}_{B2}$	a_{21}	a_{22}	$\cdots$	a_{2r}	$\cdots$	$a_{2(n+p+l)}$
$\vdots$						
$\tilde{x}_{Bi}$	a_{i1}	a_{i2}	$\cdots$	a_{ir}	$\cdots$	$a_{i(n+p+l)}$
$\vdots$						
$\tilde{x}_{Bm}$	a_{m1}	a_{m2}	$\cdots$	a_{mr}	$\cdots$	$a_{m(n+p+l)}$
$\tilde{z}\underset{\Re}{\approx}\sum_{i=1}^{m}\tilde{c}_{B_i}\tilde{x}_{B_i}$	$\tilde{z}_1\Theta\tilde{c}_1$	$\tilde{z}_2\Theta\tilde{c}_2$	$\cdots$	$\tilde{z}_r\Theta\tilde{c}_r$	$\cdots$	$\tilde{z}_{(n+p+l)}\Theta\tilde{c}_{(n+p+l)}$

where $\tilde{c}_{B1}, \tilde{c}_{B2}, ..., \tilde{c}_{Bm}$ be coefficients of $\tilde{x}_{B1}, \tilde{x}_{B2}, ..., \tilde{x}_{Bm}$ in the objective function.

Step 7 (b)

Find

$\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{=} \sum_{i=1}^{m} \tilde{c}_{B_i} a_{ij} \Theta \tilde{c}_j, \; \forall j = 1, 2, ..., (n+p+l)$

and check $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0}$ or not $\forall \; j$.

Case (i) If $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0} \quad \forall \; j$, then the following cases may arises:

Case (a) No fuzzy artificial variable is present in the basis. In such a case, a fuzzy basic feasible solution has been found. Go to Phase-II.

Case (b) At least one fuzzy artificial variable is present in the basis with positive rank. In such a case, there does not exist any fuzzy optimal solution of the given FLP problem.

Case (ii) Any fuzzy artificial variable is present in the basis with zero rank, then degenerate fuzzy basic feasible solution is available. Suppose the k^{th} fuzzy artificial variable $\tilde{x}_k \underset{\Re}{\approx} \tilde{0}$ in fuzzy basic feasible solution. Then the following two cases may arises:

Case (a) $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\neq} \tilde{0}$ for any fuzzy non-basic variable $\tilde{x}_j$, then bring $\tilde{x}_j$ to the current basis to replace fuzzy artificial variable $\tilde{x}_k$ and Go to Phase-II.

Case (b) $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\approx} \tilde{0}$ for every fuzzy non-basic variable $\tilde{x}_j$, then it can be verified that the k^{th} fuzzy constraint of the system $A"\tilde{X}" \underset{\Re}{\approx} \tilde{b}$ is redundant. In this case remove the redundant row from the original fuzzy constraints and repeat Step 7.

Case (iii) There exist at least one j such that $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{<} \tilde{0}$ proceed on to the next Step.

Step 7 (c)

If there exist one or more fuzzy variables corresponding to which $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{<} \tilde{0}$ then the fuzzy variable corresponding to which the rank of $\tilde{z}_j \Theta \tilde{c}_j$ is most negative will entering the basis. Let it be $\tilde{z}_j \Theta \tilde{c}_j$ for some $j = r$ and if $a_{ir} > 0$ for one or more values of i then compute

$$\text{minimum}\left\{ \Re\left(\frac{l_i}{a_{ir}}\right), a_{ir} > 0, i = 1, 2, ..., m \right\}$$

where, l_i is the value of i^{th} fuzzy basic variable. The fuzzy variable, corresponding to which minimum occurs, will leave the basis. Let the minimum occurs corresponding to $\tilde{x}_{Bk}$ then the common element a_{kr}, which occurs at intersection of k^{th} row and r^{th} column, is known as the leading element.

Step 7(d)

Construct the new fuzzy simplex table and calculate the new entries for the fuzzy simplex table as follows:

$\hat{a}_{kj} = \frac{a_{kj}}{a_{kr}}$ and $\hat{a}_{ij} = a_{ij} - \frac{a_{kj}}{a_{kr}} a_{ir}$ where $i = 1, 2, ..., m, i \neq k, \; j = 1, 2, ..., (n+p+l)$.

Step 7 (e)

Repeat the computational procedure from Step 7 (b) to Step 7 (d) until Case (i) or Case (ii) of Step 7 (b) is satisfied.

PHASE-II

PHASE-II is used to find the fuzzy optimal solution of the FLP problem with the help of fuzzy basic feasible solution obtained in PHASE-I. Consider the fuzzy optimal solution of PHASE-I as an initial fuzzy basic feasible solution for the original FLP problem. Assign actual coefficient to the fuzzy decision variables in the objective function and delete the columns of fuzzy artificial variables from the table obtained in PHASE-I.

Step 8

Find $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\approx} \sum^{m} \tilde{c}_{Bi} a_{ij} \Theta \tilde{c}_j, \ j = 1, 2, ..., (n+p)$.

Case (i) If $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0}, \ \forall j$ then the fuzzy basic feasible solution, obtained by using values of $\tilde{x}_{B1}, \tilde{x}_{B2}, ..., \tilde{x}_{Bm}$, is fuzzy optimal solution.

Case (ii) If there exist one or more fuzzy variables corresponding to which $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{<} \tilde{0}$ then the fuzzy variable with most negative $\tilde{z}_j \Theta \tilde{c}_j$ will enter the basis. Let it be $\tilde{z}_j \Theta \tilde{c}_j$ for some $j = r$.

Case (a) If $a_{ir} \leq 0$ then there exist fuzzy unbounded solution to the given FLP problem.

Case (b) If $a_{ir} > 0$ for one or more values of i, then repeat Step 7 (c) and Step 7(d) until Case (i) is satisfied.

Step 9

If $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\approx} \tilde{0}$, corresponding to any fuzzy nonbasic variable in the fuzzy optimal table (simplex table for which $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0}, \ \forall j$) then a fuzzy alternative solution may exist and to find it enter that fuzzy nonbasic variable into the basis and repeat once Step 7 (c) to 7 (d). The obtained fuzzy optimal solution will be a fuzzy alternative optimal solution.

7. Numerical example

In this section the proposed method is illustrated by solving a numerical example.

$$
\begin{aligned}
\textit{Maximize } \tilde{z} &\underset{\Re}{\approx} (3,5,8,13;0.7) \otimes \tilde{x}_1 \oplus (4,6,10,16;0.5) \otimes \tilde{x}_2 \\
\textit{subject to } \quad & 3\tilde{x}_1 \oplus \tilde{x}_2 \underset{\Re}{\geq} (1,2,4,7;0.7) \\
& 2\tilde{x}_1 \oplus \tilde{x}_2 \underset{\Re}{\leq} (1,3,5,6;0.9) \\
& \tilde{x}_1, \tilde{x}_2 \underset{\Re}{\geq} \tilde{0}
\end{aligned}
$$

Example 2

Solution:- Using the proposed method the fuzzy optimal solution of above FLP problem can be obtained as follow:

Step 1 Since $\tilde{x}_1, \tilde{x}_2 \underset{\Re}{\geq} \tilde{0}$, so Go to Step 2 of the proposed method.

Step 2 Since the problem is of maximization, so Go to Step 3 of the proposed method.

Step 3 Since $\Re(1,2,4,7;0.7) \underset{\Re}{\geq} 0$ and $\Re(1,3,5,6;0.9) \underset{\Re}{\geq} 0$, so Go to Step 4 of the proposed method.

Step 4 Since sign of the 1^{st} constraint and 2^{nd} constraint are $\underset{\Re}{\geq}$ and $\underset{\Re}{\leq}$ are respectively. So the sign of the constraints can be converted into $\underset{\Re}{\approx}$ by adding a fuzzy slack variable $\tilde{x}_3$ in 1^{st} constraint and by subtracting a fuzzy surplus variable $\tilde{x}_4$ from 2^{nd} constraint. Assuming the coefficients of $\tilde{x}_3$ and $\tilde{x}_4$ as zero trapezoidal fuzzy numbers in the objective function. The chosen FLP problem may be written as

$$\begin{aligned}
&\text{Maximize } \tilde{z} \underset{\Re}{\approx} (3,5,8,13;0.7)\otimes\tilde{x}_1 \oplus (4,6,10,16;0.5)\otimes\tilde{x}_2 \oplus \tilde{0}\otimes\tilde{x}_3 \oplus \tilde{0}\otimes\tilde{x}_4\\
&\text{subject to}\quad 3\tilde{x}_1 \oplus \tilde{x}_2 \Theta \tilde{x}_3 \underset{\Re}{\approx} (1,2,4,7;0.7)\\
&\qquad\qquad\quad 2\tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_4 \underset{\Re}{\approx} (1,3,5,6;0.9)\\
&\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4 \underset{\Re}{\geq} \tilde{0}
\end{aligned}$$

Step 5 Comparing the constraints of the FLP problem, obtained in Step 4, by $A'\tilde{X}' \underset{\Re}{\approx} \tilde{b}$ we have $A' = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$, $\tilde{X}' = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix}$ and $\tilde{b} = \begin{pmatrix} (1,2,4,7;\ 0.7) \\ (1,3,5,6;0.9) \end{pmatrix}$

Since neither condition of Case (i) nor Case (a) of Case (ii), discussed in Step 5 of the proposed method, is satisfied, so using Case (b) of Case (ii) there is need to add an artificial $\tilde{x}_5$ variable in the 1st constraint. After adding $\tilde{x}_5$ the above FLP problem may be written as

$$\text{Maximize } \tilde{z} \underset{\Re}{\approx} (3,5,8,13;0.7)\otimes\tilde{x}_1 \oplus (4,6,10,16;0.5)\otimes\tilde{x}_2$$

$$\begin{aligned}
&\text{subject to}\quad 3\tilde{x}_1 \oplus \tilde{x}_2 \Theta \tilde{x}_3 \oplus \tilde{x}_5 \underset{\Re}{\approx} (1,2,4,7;0.7)\\
&\qquad\qquad\quad 2\tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_4 \underset{\Re}{\approx} (1,3,5,6;0.9)\\
&\qquad\qquad\quad \tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,\tilde{x}_5 \underset{\Re}{\geq} \tilde{0}
\end{aligned}$$

Comparing the constraints of the above FLP problem by $A''\tilde{X}'' \underset{\Re}{\approx} \tilde{b}$ we have $A'' = \begin{pmatrix} 3 & 1 & -1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{pmatrix}$,

$\tilde{X}'' = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{pmatrix}$ and $\tilde{b} = \begin{pmatrix} (1,2,4,7;\ 0.7) \\ (1,3,5,6;0.9) \end{pmatrix}$

Step 6 Since in the coefficient matrix A'' the identity columns $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are corresponding to $\tilde{x}_5$ and $\tilde{x}_4$ respectively so $\tilde{x}_5$ and $\tilde{x}_4$ are the 1st t and 2nd basic variables respectively and $\tilde{X}_B = (\tilde{x}_5, \tilde{x}_4)$.

Step 7 Now fuzzy basic feasible solution of the FLP problem, obtained in Step 5, can be obtained by using PHASE-I.

PHASE-I

The FLP problem, obtained in Step 5, with auxiliary objective function may be written as:

$$\begin{aligned} &\textit{Maximize } \tilde{z} \underset{\Re}{\approx} (-1,-1,-1,-1;1) \otimes \tilde{x}_5 \\ &\textit{subject to} \quad 3\tilde{x}_1 \oplus \tilde{x}_2 \Theta \tilde{x}_3 \oplus \tilde{x}_5 \underset{\Re}{\approx} (1,2,4,7;0.7) \\ &\qquad 2\tilde{x}_1 \oplus \tilde{x}_2 \oplus \tilde{x}_4 \underset{\Re}{\approx} (1,3,5,6;0.9) \\ &\qquad \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5 \underset{\Re}{\geq} \tilde{0} \end{aligned}$$

Using Step 7 (a) of the proposed method the fuzzy simplex table for the above problem is

Table 2.

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$
$\tilde{x}_5 \underset{\Re}{\approx} (1,2,4,7;0.7)$	3	1	-1	0	1
$\tilde{x}_4 \underset{\Re}{\approx} (1,3,5,6;\ 0.9)$	2	1	0	1	0
$\tilde{z} \underset{\Re}{\approx} (-7,-4,-2,-1;0.7)$	$(-3,-3,-3,-3;1)$	$(-1,-1,-1,-1;1)$	$(1,1,1,1;1)$	$\tilde{0}$	$\tilde{0}$

Since $\Re(-3,-3,-3,-3;1)$ and $\Re(-1,-1,-1,-1;1) < 0$, so Case (iii) of Step 7 (b) of the proposed method is satisfied. Using Step 7 (c) of the proposed method, fuzzy variable $\tilde{x}_1$ will enter in the basis and the fuzzy variable $\tilde{x}_5$ will leave the basis. Using Step 7 (d) of the proposed method new fuzzy simplex table is

Table 3.

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$
$\tilde{x}_1 \underset{\Re}{\approx} \left(\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{7}{3};0.7\right)$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$
$\tilde{x}_4 \underset{\Re}{\approx} \left(-\frac{11}{3},\frac{1}{3},\frac{11}{3},\frac{16}{3};0.7\right)$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{2}{3}$
$\tilde{z} \underset{\Re}{\approx} \tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$(1,1,1,1;1)$

Since $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0} \quad \forall\ j$, and no fuzzy artificial variables is present in the basis. So a fuzzy basic feasible solution has been found and Go to PHASE-II. So fuzzy variables $\tilde{x}_1$ and $\tilde{x}_4$ are the fuzzy basic variables for the original problem.

PHASE-II

Considering the fuzzy optimal solution of PHASE-I as an initial fuzzy basic feasible solution for the original FLP problem, assigning actual coefficient to the fuzzy decision variables in the objective function and deleting the columns of fuzzy artificial variables from the table obtained in PHASE-I. The obtained fuzzy simplex table is:

Table 4.

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
$\tilde{x}_1 \underset{\Re}{\approx} \left(\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{7}{3};0.7\right)$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0
$\tilde{x}_4 \underset{\Re}{\approx} \left(-\frac{11}{3},\frac{1}{3},\frac{11}{3},\frac{16}{3};0.7\right)$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\tilde{z} \underset{\Re}{\approx} \left(1,\frac{10}{3},\frac{32}{3},\frac{91}{3};0.7\right)$	$\tilde{0}$	$\left(-15,-\frac{25}{3},-\frac{10}{3},\frac{1}{3};0.5\right)$	$\left(-\frac{13}{3},-\frac{8}{3},-\frac{5}{3},-1;0.5\right)$	$\tilde{0}$

Since $\Re\left(-15,-\frac{25}{3},-\frac{10}{3},\frac{1}{3};0.5\right)<0$ and $\Re\left(-\frac{13}{3},-\frac{8}{3},-\frac{5}{3},-1;0.5\right)<0$ so using Step 8 of the proposed method, fuzzy variable $\tilde{x}_2$ will enter the basis and the fuzzy variable $\tilde{x}_1$ will leave the basis. Using Step 7 (d) of the proposed method new fuzzy simplex table is

Table 5.

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
$\tilde{x}_2 \underset{\Re}{\approx} (1,2,4,7;0.7)$	3	1	-1	0
$\tilde{x}_3 \underset{\Re}{\approx} (-6,-1,40,112;\ 0.7)$	-2	0	1	1
$\tilde{z} \underset{\Re}{\approx} (4,12,40,112;0.7)$	$(-1,10,25,45;0.5)$	$\tilde{0}$	$(-16,-10,-6,-4;0.7)$	$\tilde{0}$

Since $\Re(-16,-10,-6,-4;0.7)$ and $\Re(-1,10,25,45;0.5)<0$, so using Step 8 of the proposed method, fuzzy variable $\tilde{x}_4$ will enter the basis and the fuzzy variable $\tilde{x}_3$ will leave the basis. Using Step 7 (d) of the proposed method new fuzzy simplex table is

Table 6.

$\tilde{X}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$
$\tilde{x}_2 \underset{\Re}{\approx} (-5,1,7,12;0.7)$	2	1	0	1
$\tilde{x}_4 \underset{\Re}{\approx} (-6,-1,3,5;\ 0.7)$	-1	0	1	1
$\tilde{z} \underset{\Re}{\approx} (-20,6,70,192;0.7)$	$(-5,4,15,29;0.5)$	$\tilde{0}$	$\tilde{0}$	$(4,6,10,16;0.5)$

Since $\tilde{z}_j \Theta \tilde{c}_j \underset{\Re}{\geq} \tilde{0} \quad \forall \ j$,, so using Case (i) of Step 8, there is no entering variable. Therefore, the obtained fuzzy basic feasible solution is fuzzy optimal solution. Hence $\tilde{x}_1 \underset{\Re}{\approx} \tilde{0}, \tilde{x}_2 \underset{\Re}{\approx} (-5,1,7,12;0.7)$ and *Maximize* $\tilde{z} \underset{\Re}{\approx} (-20,6,70,192;0.7)$.

8. Results and discussion

In this section the advantages of the proposed method, for solving a special type of FLP problem, are discussed.

8.1. Advantages of the proposed method

In this subsection the advantages of the proposed method, for solving generalized FLP problems, are discussed.

8.1.1. Disadvantages of the normalization process

In most of the papers the generalized fuzzy numbers are converted into normal fuzzy numbers through normalization process [27] and then normal fuzzy numbers are used to solve the real life problems. Kaufmann and Gupta [27] pointed out that there is a serious disadvantage of the normalization process. Basically we have transformed a measurement of an objective value to a valuation of a subjective value, which results in the loss of information. Although this procedure is mathematically correct, it decreases the amount of information that is available in the original data, and we should avoid it.

To explain the disadvantages of the normalization process the same example is solved, using the proposed method, with and without normalization process and then the obtained results are discussed as follows:

8.1.1.1. Results with normalization process

If all the values of the parameters, used in numerical example solved in Section 7, is first normalized and then the problem is solved by using the proposed method then the fuzzy optimal value is $\tilde{z} \underset{\Re}{\approx} (-20,6,70,192;1)$. The membership function for the obtained result is shown in Fig. 1. From the Figure 1 it is clear that

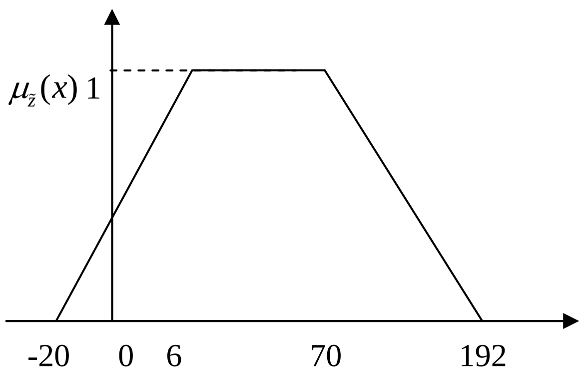

Figure 1. Membership function of $\tilde{z}$

(i) According to decision maker the total profit will be greater than Rs 4 and less than Rs 192.

(ii) Decision maker is 100% in favour that the total profit will be greater than or equal to Rs 6 and less than or equal to Rs 70.

(iii) The percentage of the favourness of the decision maker for the remaining values of total profit can be obtained as follows:

Let x represents the value of total profit then the percentage of the favourness of the decision maker for $x = \mu_{\tilde{z}}(x) \times 100$, where $\mu_{\tilde{A}}(x) = \begin{cases} 0 & , -\infty < x < -20 \\ \frac{(x+20)}{26} & , -20 \le x \le 6 \\ 1 & , 6 \le x \le 70 \\ \frac{(x-192)}{-122} & , 70 \le x \le 192 \\ 0 & , 192 < x < -\infty \end{cases}$

8.1.1.2. Results without normalization process

If the values of the parameters of the same example is not normalized and then the problem is solved (as in section 7) by using the proposed method then the fuzzy optimal value is $\tilde{z} \underset{\Re}{\approx} (-20, 6, 70, 192; 0.7)$. The membership function for the obtained result is shown in Fig. 2. From the Figure 2 it is clear that

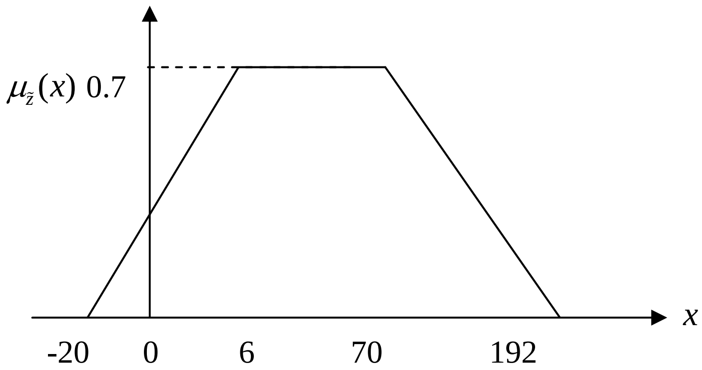

Figure 2. Membership function of $\tilde{z}$

(i) According to decision maker the total profit will be greater than Rs 4 and less than Rs 192.

(ii) Decision maker is 70% in favour that the total profit will be greater than or equal to Rs 6 and less than or equal to Rs 70.

(iii) The percentage of the favourness of the decision maker for the remaining values of total profit can be obtained as follows:

Let x represents the value of total profit then the percentage of the favourness of the decision maker for $x = \mu_{\tilde{z}}(x) \times 100$, where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , -\infty < x < -20 \\ \frac{0.7(x+20)}{26} & , -20 \le x \le 6 \\ 0.7 & , 6 \le x \le 70 \\ \frac{0.7(x-192)}{-122} & , 70 \le x \le 192 \\ 0 & , 192 < x < -\infty \end{cases}$$

It is obvious from the results explained in sections 8.1.1.1 and 8.1.1.2 that according to decision maker the range of objective function is same in both cases i.e., total profit will be greater than $Rs\ -20$ and less than $Rs\ 192$ but due to normalization process the actual percentage of favourness for different values of total profit is changed. For example: Results with normalization process represents that the favourness of decision maker about the value of total profit lying between $Rs\ 6$ and $Rs\ 70$ is 100% while without normalization process the percentage of favourness for the same range is 70%.

Similarly it can be found that the percentage of favourness for the same values of total profit are different in both cases i.e., using the normalization process the actual information is lost.

9. Conclusion

In this paper, two phase method is introduced for solving a particular type of FLP problem occurring in real life problems and also the advantages of proposed method are discussed. The proposed method is a direct extension of classical method so it is very easy to understand and apply the proposed method to find the fuzzy optimal solution of FLP problems. From the results it can be concluded that it is better to represent the parameters as generalized fuzzy numbers instead of normal fuzzy numbers. In future the proposed work can be extended to proposed new methods for finding the fuzzy optimal solution of fuzzy linear programming programming problems by representing all the parameters as vague sets [28, 29].

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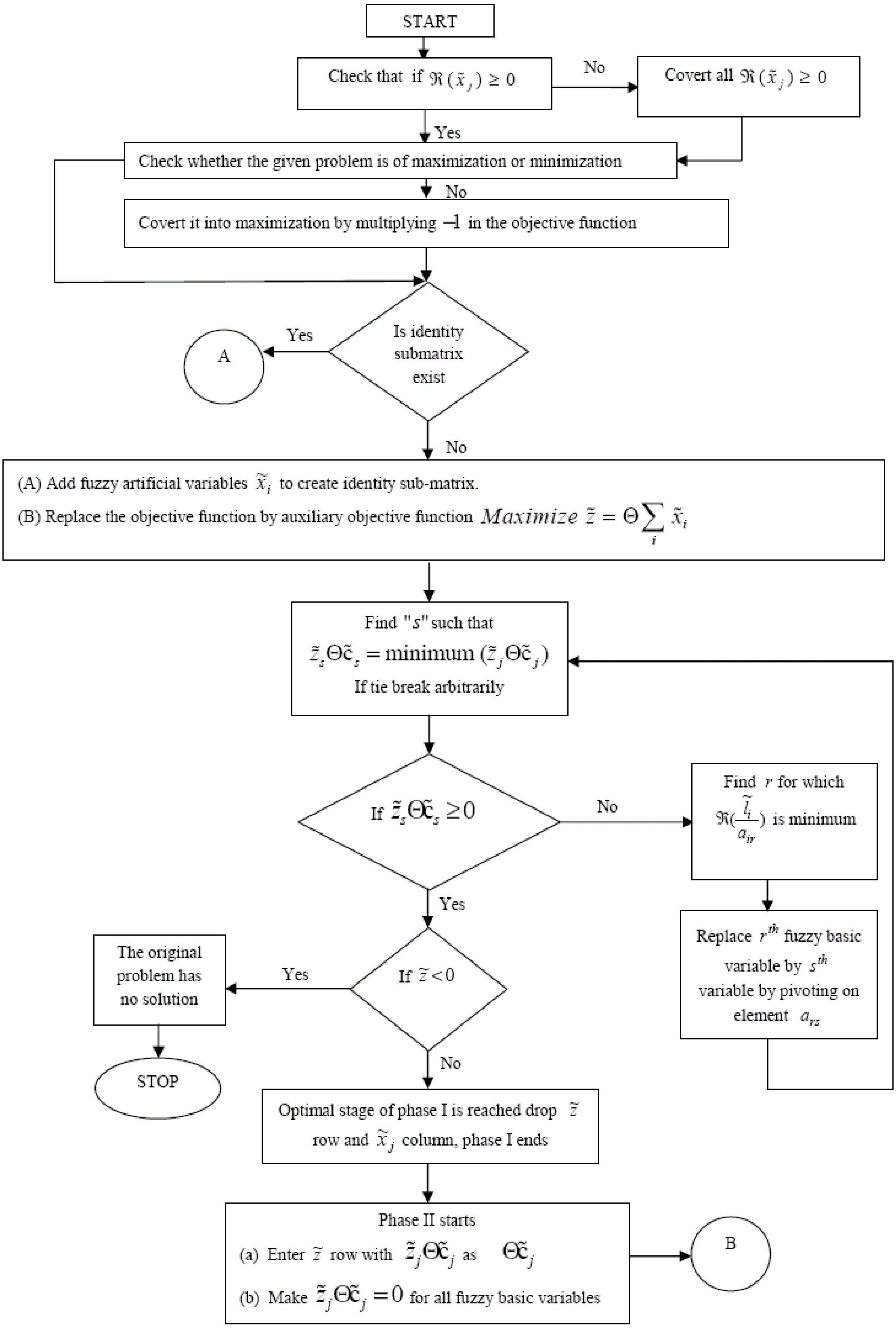
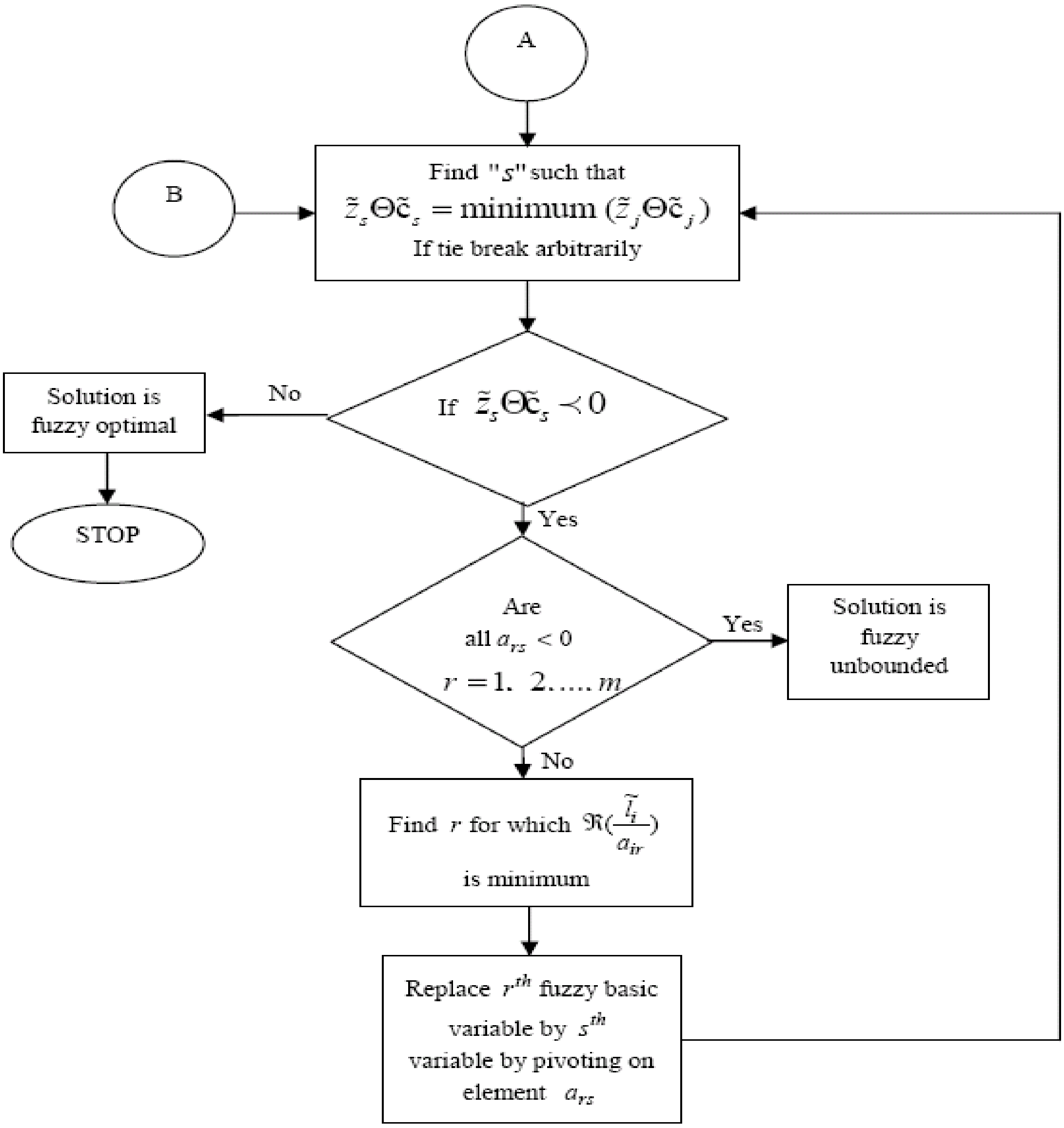
Appendix

Flow Chart for two phase method is given as follows:


START
Check that if $\Re(\tilde{x}_j) \geq 0$
No
Covert all $\Re(\tilde{x}_j) \geq 0$
Yes
Check whether the given problem is of maximization or minimization
No
Covert it into maximization by multiplying -1 in the objective function
Is identity submatrix exist
Yes
A
No
(A) Add fuzzy artificial variables $\tilde{x}_i$ to create identity sub-matrix.
(B) Replace the objective function by auxiliary objective function $Maximize\ \tilde{z} = \Theta \sum_i \tilde{x}_i$
Find "s" such that
$\tilde{z}_s \Theta \tilde{c}_s = \text{minimum } (\tilde{z}_j \Theta \tilde{c}_j)$
If tie break arbitrarily
If $\tilde{z}_s \Theta \tilde{c}_s \geq 0$
No
Find r for which
$\Re(\frac{\tilde{l}_i}{a_{ir}})$ is minimum
Replace r^{th} fuzzy basic variable by s^{th} variable by pivoting on element a_{rs}
Yes
If $\tilde{z} < 0$
Yes
The original problem has no solution
STOP
No
Optimal stage of phase I is reached drop $\tilde{z}$ row and $\tilde{x}_j$ column, phase I ends
Phase II starts
(a) Enter $\tilde{z}$ row with $\tilde{z}_j \Theta \tilde{c}_j$ as $\Theta \tilde{c}_j$
(b) Make $\tilde{z}_j \Theta \tilde{c}_j = 0$ for all fuzzy basic variables
B



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