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A Fitted Galerkin Method for Singularly Perturbed Differential Equations with Layer Behaviour

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Abstract: In this paper, we have presented a fitted Galerkin method for singularly perturbed differential equations with layer behaviour. We have introduced a fitting factor in the Galerkin difference scheme which takes care of the rapid changes occur that in the boundary layer. This fitting factor is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the tridiagonal system of the fitted Galerkin method. The existence and uniqueness of the discrete problem along with stability estimates are discussed. Also we have discussed the convergence of the method. Maximum absolute errors in numerical results are presented to illustrate the proposed method.

Keywords: Singularly perturbed two-point boundary value problem; boundary layer; Taylor series; Galerkin method; maximum absolute error.

1. Introduction

During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. Typically, these problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, Wentzel, Kramers and Brillouin (WKB) problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large peclet numbers, etc. The numerical treatment of singular perturbation problems has always been far from trivial, because of the boundary layer behavior of the solutions. However, the area of singular perturbations is a field of increasing interest to applied mathematicians. Much progress has been made recently in developing finite element methods for solving singular perturbation problems. Several authors Eckhaus [4], Natesan and Ramanujam [10], Valanarasu and Ramanujam [14] have investigated solving singular perturbation problems by numerically constructing asymptotic solutions. The general motivation is to provide simpler efficient computational techniques to solve singular perturbation problems. A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these, we mention Bawa [1], Bellman [2], Bender [3], Hemker et. al. [5], Kadalbajoo, Reddy [6], Kadalbajoo and Patidar [7], Kevorkian and Cole [9], Nayfeh [11], O' Malley [12], Ramos et. al. [13], Van Dyke [15] and Vigo-Aguiar, Natesan [16]. Kasi Viswanadham et. al. [8] presented a numeri-

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cal solution of fifth order boundary value problems using sixth order B-Splines.
There is a wide variety of asymptotic expansion methods available for solving the problems of the above type. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions, which are not routine exercises but require skill, insight, and experimentations. In view of the wealth of the literature available on singular perturbation problems and in view of the specialized skills and experience that experts in the field deem necessary, one can raise the question whether there may be other ways to attack these problems, ways that are easy to use and ready for computer implementation, ways that are more accessible to the practicing engineers or applied mathematicians. The spline technique is one such tool to reach these goals in an optimum way.

The fitted technique is one such tool to reach these goals in an optimum way. There are two possibilities to obtain small truncation error inside the boundary layer(s). The first is to choose a fine mesh there, whereas the second one is to choose a difference formula reflecting the behaviour of the solution(s) inside the boundary layer(s). Present work deals with the second approach. In this paper, we introduce fitting factor $\sigma(\rho)$ to the term contains perturbation parameter ε affecting the highest derivative. This fitting factor is determined in such a way that the truncation error of the corresponding scheme for the boundary layer function(s), in the case of constant coefficients, should be equal to zero. This procedure is known as the exponential fitting or the introducing of artificial viscosity.

2. Numerical Method

2.1. Left-End Boundary Layer Problems

Consider a linearly singularly perturbed two point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1] \quad (1)$$

with the boundary conditions

$$y(0) = \alpha \quad (2a)$$

and $y(1) = \beta$ (2b)

We assume that *a(x), b(x)* and *f(x)* are sufficiently continuously differentiable functions in
[0, 1]. Further more, we assume that $b(x) \leq 0$, $a(x) \geq M > 0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution *y(x)* which in general, displays a boundary layer of width O(ε) at $x = 0$ for small values of ε.
From the theory of singular perturbation s it is known that the solution of (1) - (2) is of the form (O'Malley [12])

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{\int_0^x \left(\frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)}\right)dx} + O(\varepsilon) \quad (3)$$

Where $y_0(x)$ is the solution of

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \quad y_0(1) = \beta \quad (4)$$

By taking Taylor's series expansion for a(x) and b(x) about the point '0' and restricting to their first terms, (3) becomes,

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x} + O(\varepsilon) \quad (5)$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let

0=$x_1, x_2, ...x_N$ =1 be the mesh points.
Then we have $x_i = ih : i = 0, 1, 2, \ldots\ldots, N$.
From (5), we have

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)x_i} + O(\varepsilon)$$

i.e.,

$$y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)ih} + O(\varepsilon)$$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)i\rho} \quad (6)$$

where $\rho = \dfrac{h}{\varepsilon}$

Now we consider the difference scheme by Galerkin method as follows:
Select a set of basis functions $\phi_j(x),\ j = 0,1,2,......N$ which will define an interpolation scheme for the approximate solution over a grid of points $a = x_0 < x_1 < .... < x_{N+1} = b$.

For simplicity we use piecewise Lagrange polynomials $l_i(x)$ of first degree as the basis functions $\phi_j(x)$. These interpolating polynomials are

$$l_0(\xi) = \frac{\xi - \xi_1}{\xi_0 - \xi_1} = \frac{(1-\xi)}{2} \quad (7)$$

$$l_1(\xi) = \frac{\xi - \xi_0}{\xi_1 - \xi_0} = \frac{(1+\xi)}{2} \quad (8)$$

in local element coordinates $-1 \le \xi \le 1$.

The N nodal values of the approximate solution y at the interior nodes $x_1, x_2, ..., x_N$ are determined using this basis. The given boundary conditions determine the value of *y(x)* at the end nodes x_0 and x_{N+1}.
The Galerkin method is now employed to obtain the integral equation's (Fletcher, 1984), we have

$$(\varepsilon y''(x) + a(x)y'(x) + b(x)y(x), \phi_j) = (f(x), \phi_j) \quad \text{for} \quad j = 1,2,......N \quad (9)$$

Since y is sum of piecewise linear Lagrange polynomials, the second order derivatives appearing in Eq.(9) vanish except at the element boundaries x_i, where they become infinite.
By integration by parts, (9) becomes

$$-\left(\varepsilon\frac{dy}{dx}, \frac{d\phi_j}{dx}\right) + \left(a(x)\frac{dy}{dx} + b(x)y, \phi_j\right) + \left(\varepsilon\frac{dy}{dx}\phi_j\right)_a^b = (\mathrm{f(x)},\ \phi_j) \quad (10)$$

The substitution of trial function $y(x) = \phi_0(x) + \sum_{i=1}^{N} y_i\phi_j(x)$ into the integral equation (10), we have

$$\sum_{i=1}^{N} y_i\left(\varepsilon\frac{d\phi_i}{dx}, \frac{d\phi_j}{dx}\right) - \sum_{i=1}^{N} y_i\left(a(x)\frac{d\phi_i}{dx} + b(x)\phi_i, \phi_j\right) = -\alpha\left(\frac{dl_0}{dx}, \frac{d\phi_j}{dx}\right) - \beta\left(\frac{dl_{N+1}}{dx}, \frac{d\phi_j}{dx}\right)$$
$$+\alpha\left(a(x)\frac{dl_0}{dx} + b(x)l_0, \phi_j\right) + \beta\left(a(x)\frac{dl_{N+1}}{dx} + b(x)l_{N+1}, \phi_j\right) + \quad (11)$$
$$\left(\varepsilon\frac{dy}{dx}\phi_j\right)_a^b - (f(x), \phi_j)$$

for j =1, 2, …, N

It can be observed that all quantities on the right side of Eq. (11) can be computed from known boundary data to obtain N equations in the N unknown values y_i at the interior nodes.

The integrals in Eq. (11) can be solved by taking advantage of local coordinate (ξ) system. Since

$$x = \frac{(b-a)}{2}\xi + \frac{(b+a)}{2} = \frac{h}{2}\xi + \frac{(b+a)}{2}$$

$$\xi = \frac{2}{h}\left(x - \frac{(b+a)}{2}\right) \Rightarrow \frac{d\xi}{dx} = \frac{2}{h},$$

we have by simple integration,

$$\int_a^b \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \int_{-1}^{1}\frac{dl_0}{d\xi}\frac{d\xi}{dx}\frac{dl_1}{d\xi}\frac{d\xi}{dx}\frac{h}{2}d\xi = \frac{-1}{h}, \text{ for } i = j-1,$$

$$\int_a^b \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \int_{-1}^{1}\frac{dl_1}{d\xi}\frac{d\xi}{dx}\frac{dl_0}{d\xi}\frac{d\xi}{dx}\frac{h}{2}d\xi = \frac{-1}{h}, \text{ for } i = j+1,$$

$$\int_a^b \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = \int_{-1}^{1}\frac{dl_1}{d\xi}\frac{d\xi}{dx}\frac{dl_1}{d\xi}\frac{d\xi}{dx}\frac{h}{2}d\xi = \frac{1}{h}, \text{ for } i = j,$$

$$\int_a^b \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = 0 \quad \text{for} |i - j| > 1.$$

since *a(x)*,*b(x)* and *f(x)* are constants, the integral equation (11) give, for a typical internal node *j*,

since *a(x)*,*b(x)* and *f(x)* are constants, the integral equation (11) give, for a typical internal node *j*,

$$y_{j-1}\left(\frac{-\varepsilon}{h} + \frac{a}{2} - \frac{bh}{6}\right) + y_j\left(\frac{2\varepsilon}{h} - \frac{2bh}{3}\right) + y_{j+1}\left(\frac{-\varepsilon}{h} - \frac{a}{2} - \frac{bh}{6}\right) = -f_j h \tag{12}$$

the Eq. (12) when rearranged gives the following system of difference equations and we call it as Galerkin difference scheme

$$\varepsilon\left(\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}\right) + a\left(\frac{y_{j+1} - y_{j-1}}{2h}\right) + b\left(\frac{y_{j-1} + 4y_j + y_{j+1}}{6}\right) = f_j \text{ ; for } 1 \le j \le N-1$$

Now introduce a fitting factor in the Galerkin difference scheme, we get

$$\varepsilon\sigma(\rho)\left(\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}\right) + a\left(\frac{y_{j+1} - y_{j-1}}{2h}\right) + b\left(\frac{y_{j-1} + 4y_j + y_{j+1}}{6}\right) = f_j \tag{13}$$

for $1 \le j \le N-1$.

with $y_0 = \alpha$; $y_N = \beta$; where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of Eq. (13) converges uniformly to the solution of (1), (2) & (3).

Multiplying (13) by *h* and taking the limit as $h \to 0$, we get

$$\lim_{h\to 0}\left(\frac{\sigma(\rho)}{\rho}\left(y_{i+1} - 2y_i + y_{i-1}\right) + \frac{1}{2}a(ih)\left(y_{i+1} - y_{i-1}\right)\right) = 0 \text{ if } f(x_i) - b(x_i)y_i \text{ is bounded.}$$

$$\therefore \lim_{h\to 0}\left(\frac{\sigma(\rho)}{\rho}\left(y(ih+h) - 2y(ih) + y(ih-h)\right) + \frac{1}{2}a(ih)\left(y(ih+h) - y(ih-h)\right)\right) = 0 \tag{14}$$

Substituting (6) in (14) and simplifying, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho} = \frac{1}{2}a(0)Coth\left[\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right] \tag{15}$$

$$\sigma = \frac{\rho}{2}a(0)Coth\left[\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\frac{\rho}{2}\right] \tag{16}$$

which is a constant fitting factor.
From eq.(13) we have

$$y_{j-1}\left(\frac{\varepsilon\sigma}{h^2}-\frac{a}{2h}+\frac{b}{6}\right)-y_j\left(\frac{2\varepsilon\sigma}{h^2}-\frac{2b}{3}\right)+y_{j+1}\left(\frac{\varepsilon\sigma}{h^2}+\frac{a}{2h}+\frac{b}{6}\right)=f_j \quad (17)$$

for $j=1,2,.....,N-1$; where the fitting factor σ is given by (16).
The equation (17) can be written as a three term recurrence relation:

$$E_j y_{j-1}-F_j y_i+G_j y_{j+1}=H_j\ ;\ j=1,2,...N-1 \quad (18)$$

where

$$E_j=\left(\frac{\varepsilon\sigma}{h^2}-\frac{a}{2h}+\frac{b}{6}\right)$$

$$F_j=\left(\frac{2\varepsilon\sigma}{h^2}-\frac{2b}{3}\right)$$

$$G_j=\left(\frac{\varepsilon\sigma}{h^2}+\frac{a}{2h}+\frac{b}{6}\right)$$

$$H_j=f_j$$

This gives us the tridiagonal system which can be solved easily by Thomas Algorithm.

2.2. Right-End Boundary Layer Problems

We discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form (1) with (2a) and (2b)
where ε is a small positive parameter ($0<\varepsilon<<1$) and α,β are known constants.
We assume that *a(x), b(x)* and *f(x)* are sufficiently continuously differentiable functions in [0, 1]. Further more, we assume that *a(x)* ≤ M <0 throughout the interval [0,1], where M is some negative constant. Under these assumptions, (1) has a unique solution y(x) which in general, displays a boundary layer of width O(ε) at x =1 for small values of ε.
From the theory of singular perturbation s it is known that the solution of (1)-(2) is of the form (O'Malley [12])

$$y(x)=y_0(x)+\frac{a(1)}{a(x)}(\beta-y_0(1))e^{\int_x^1\left(\frac{a(x)}{\varepsilon}-\frac{b(x)}{a(x)}\right)dx}+O(\varepsilon) \quad (19)$$

Where $y_0(x)$ is the solution of

$$a(x)y_0'(x)+b(x)y_0(x)=f(x),\ \ \mathrm{y}_0(0)=\alpha \quad (20)$$

By taking Taylor's series expansion for *a(x)* and *b(x)* about the point '1' and restricting to their first terms, (19) becomes,

$$y(x)=y_0(x)+(\beta-y_0(1))e^{-\left(\frac{a(1)}{\varepsilon}-\frac{b(1)}{a(1)}\right)(1-x)}+O(\varepsilon) \quad (21)$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h.
Let 0=$x_1,x_2,.......x_N$=1 be the mesh points. Then we have $x_i=ih:i=0,1,2,$…….N.
From (21), we have

i.e., $y(ih) = y_0(ih) + (\beta - y_0(1))e^{-\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-ih)} + O(\varepsilon)$ therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{-\left(\frac{a^2(1)-\varepsilon b(1)}{a(1)}\right)(\frac{1}{\varepsilon} - i\rho)} \tag{22}$$

Where $\rho = \dfrac{h}{\varepsilon}$

Now consider the difference scheme (13) and we will get the fitting factor as

$$\sigma = \frac{\rho}{2} a(0) Coth\left[\left(\frac{a^2(1) - \varepsilon b(1)}{a(1)}\right)\frac{\rho}{2}\right] \tag{23}$$

Then from (13) we have the difference scheme (17) where fitting factor is given by (23) and then the three term recurrence relation (18) which gives tri diagonal system which can be solved easily by Thomas Algorithm.

3. Stability and convergence analysis

Theorem 1. Under the assumptions $\varepsilon > 0$, $a(x) \ge M > 0$ and $b(x) < 0$, $\forall x \in [0,1]$, the solution to the system of the difference equations (18), together with the given boundary conditions exists, is unique and satisfies

$$\|y\|_{h,\infty} \le 2M^{-1}\|f\|_{h,\infty} + (|\alpha| + |\beta|)$$

where $\|\cdot\|_{h,\infty}$ is the discrete $l_\infty - norm$, given by $\|x\|_{h,\infty} = \max_{0\le i\le N}\{|x_i|\}$.

Proof. Let $L_h(.)$ denote the difference operator on left hand side of Eq. (12) and w_i be any mesh function satisfying

$L_h(w_i) = f_i$

By rearranging the difference scheme (12) and using non-negativity of the coefficients E_i, F_i and G_i, we obtain

$F_i\,|w_i| \le |H_i| + E_i\,|w_{i-1}| + G_i\,|w_{i+1}|$

Now using the assumption ε>0 and $a_i \ge M$, the definition of l_∞-norm and manipulating the above inequality, we obtain

$$\sigma\,\varepsilon \frac{(|w_{i+1}| - 2|w_i| + |w_{i-1}|)}{h^2} + a_i\left(\frac{|w_{i+1}| - |w_{i-1}|}{2h}\right) + \frac{b_i}{6}(|w_{i-1}| + 4|w_i| + |w_{i+1}|) + |f_i| \ge 0 \tag{24}$$

To prove the uniqueness and existence, let $\{u_i\}, \{v_i\}$ be two sets of solution of the difference equation (17) satisfying boundary conditions. Then

$w_i = u_i - v_i$ satisfies $L_h(w_i) = f_i$ where $f_i = 0$ and $w_0 = w_N = 0$.

Summing (24) over i = 1, 2, ……, N-1, we obtain

$$-\sigma\varepsilon\frac{|w_1|}{h^2} - \sigma\varepsilon\frac{|w_{N-1}|}{h^2} - \|a\|_{h,\infty}\frac{|w_1|}{2h} + \frac{5b}{6}\sum_{i=1}^{N-1}|w_i| \ge 0 \tag{25}$$

Since $\varepsilon > 0$, $\|a\|_{h,\infty} \ge 0, b_i < 0$ and $|w_i| \ge 0$ $\forall i, i = 1,2,......, N-1$,

therefore for inequality (25) to hold, we must have $w_i = 0$ $\forall i,\ i = 1,2,.....N-1$. This implies the uniqueness of the solution of the tridiagonal system of difference equations

(18). For linear equations, the existence is implied by uniqueness. Now to establish the estimate, let $w_i = y_i - l_i$, where y_i satisfies difference equations (17), the boundary conditions and
$l_i = (1-ih)\alpha + (ih)\beta$, then $w_0 = w_N = 0$,

and w_i, $i = 1,2,...N-1$, $L_h(w_i) = f_i$ Now let
$|w_n| = \|w\|_{h,\infty} \geq |w_i|, i = 0,1,......., N.$

Then summing (24) from $i = n$ to N-1 and using the assumption on *a(x),* which gives

$$-\varepsilon\frac{(|w_n|-|w_{n-1}|)}{h^2} - \varepsilon\frac{|w_{N-1}|}{h^2} - M\frac{|w_{n-1}|}{2h} - M\frac{|w_n|}{2h} + M\frac{|w_{N-1}|}{2h} + \frac{b}{6}|w_{n-1}| + \frac{5b}{6}\sum_{i=n}^{N-1}|w_i| + \sum_{i=n}^{N-1}|f_i| \geq 0 \quad (26)$$

Inequality (26), together with the condition on *b(x)* implies that
$\frac{M}{2}|w_n| \leq h\sum_{i=n}^{N-1}|f_i| \leq h\sum_{i=0}^{N}|f_i| \leq \|f\|_{h,\infty}$, i.e., we have

$$|w_n| \leq 2M^{-1}\|f\|_{h,\infty} \quad (27)$$

Also, we have $y_i = w_i + l_i$,

$$\begin{aligned}\|y\|_{h,\infty} &= \max_{0\leq i\leq N}\{|y_i|\}\\ &\leq \|w\|_{h,\infty} + \|l\|_{h,\infty} \quad (28)\\ &\leq |w_n| + \|l\|_{h,\infty}.\end{aligned}$$

Now to complete the estimate, we have to find out the bound on l_i

$$\begin{aligned}\|l\|_{h,\infty} &= \max_{0\leq i\leq N}\{|l_i|\}\\ &\leq \max_{0\leq i\leq N}\{|(1-ih)||\alpha| + |ih||\beta|\} \text{ i.e., we}\\ &\leq \max_{0\leq i\leq N}\{(1-ih)|\alpha| + (ih)|\beta|\},\end{aligned}$$

have

$$\| l \|_{h,\infty} \leq |\alpha| + |\beta|. \quad (29)$$

From Eqs. (27) – (29), we obtain the estimate
$\|y\|_{h,\infty} \leq 2M^{-1}\|f\|_{h,\infty} + (|\alpha| + |\beta|).$

This theorem implies that the solution to the system of the difference equations (18) are uniformly bounded, independent of mesh size h and the perturbation parameter ε. Thus the scheme is stable for all step sizes.

Corollary 1. Under the conditions for theorem 1, the error $e_i = y(x_i) - y_i$ between the solution *y(x)* of the continues problem and the solution y_i of the discretized problem, with boundary conditions, satisfies the estimate

$\|e\|_{h,\infty} \leq 2M^{-1}\|\tau\|_{h,\infty}$, where $|\tau_i| \leq \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{6}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{6}|y^{(2)}(x)|\right\}$

Proof. Truncation error τ_i is given by

$$\tau_i = \sigma\varepsilon\left\{\left(\frac{y_{i+1} - y_i + y_{i-1}}{h^2}\right) - y_i''\right\} + a(x)\left\{\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - y_i' + b(x)\left\{\left(\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right) - y_i\right\}\right\}$$

$$|\tau_i| \leq \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{6}|y^{(3)}(x)|\right\} + \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{12}|y^{(4)}(x)|\right\} + \max_{x_{i-1}\leq x\leq x_{i+1}}\left\{\frac{h^2}{6}|y^{(2)}(x)|\right\}$$

One can easily show that the error e_i, satisfies

$$L_h(e(x_i)) = L_h(y(x_i)) - L_h(y_i) = \tau_i,\ i = 1,2,...,N-1 \text{ And } e_0 = e_N = 0.$$

Then Theorem 1 implies that

$$\|e\|_{h,\infty} \le 2M^{-1}\|\tau\|_{h,\infty} \tag{30}$$

The estimate (30) establishes the convergence of the difference scheme for the fixed values of the parameter ε.

Theorem 2. Under the assumptions $\varepsilon > 0$, $a(x) \le M < 0$ and $b(x) < 0, \forall x \in [0,1]$, the solution to the system of the difference equations (18), together with the given boundary conditions exists, is unique and satisfies

$$\|y\|_{h,\infty} \le 2M^{-1}\|f\|_{h,\infty} + (|\alpha| + |\beta|).$$

The proof of estimate can be done on similar lines as we did in theorem 1.

4. Numerical Examples

To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer and two linear singular perturbation problems with right-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The numerical solutions are compared with the exact solutions and maximum absolute errors with and without fitting factor are presented to support the given method.

Example 1. Consider the following homogeneous singular perturbation problem from Bender and Orszag [3]

$\varepsilon y''(x) + y'(x) - y(x) = 0$; $x \in [0,1]$ with $y(0) = 1$ and $y(1) = 1$. Clearly this problem has a boundary layer at x = 0. i.e., at the left end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{[(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}]}{[e^{m_2} - e^{m_1}]} \quad \text{Where}$$

$m_1 = (-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and

$m_2 = (-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$

The maximum absolute errors are presented in tables 1 for different values of ε.

Example 2. Now consider the following non-homogeneous singular perturbation problem

$\varepsilon y''(x) + (1+\varepsilon)y'(x) + y(x) = 0$; $x \in [0,1]$

with $y(0) = 0$ and $y(1) = 1$.

Clearly this problem has a boundary layer at x = 0. The exact solution is given by

$$y(\mathrm{x}) = \frac{\left(e^{-x} - e^{-x/\varepsilon}\right)}{\left(e^{-1} - e^{-1/\varepsilon}\right)}$$

The maximum absolute errors are presented in tables 2 for different values of ε.

Example 3. Consider the following singular perturbation problem

$\varepsilon y''(x) + y'(x) = 2$; $x \in [0,1]$ with $y(0) = 0$ and $y(1) = 1$. The exact solution is given by

$$y(x) = 2x + \frac{1 - e^{-\left(x/\varepsilon\right)}}{e^{-\left(1/\varepsilon\right)} - 1}.$$

The maximum absolute errors with fitting factor are presented in tables 3 for different values of ε and the maximum absolute errors without fitting factor are presented in table 4 for comparison.

Example 4. Consider the following singular perturbation problem

$\varepsilon y''(x) - y'(x) = 0$; $x \in [0,1]$ with $y(0) = 1$ and $y(1) = 0$. Clearly, this problem has a boundary layer at x=1. i.e., at the right end of the underlying interval.

The exact solution is given by

$$y(x) = \frac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$$

The maximum absolute errors with fitting factor are presented in tables 5 for different values of ε and the maximum absolute errors without fitting factor are presented in table 6 for comparison.

Example 5. Now we consider the following singular perturbation problem $\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0$; $x \in [0,1]$ with $y(0) = 1+\exp(-(1+\varepsilon)/\varepsilon)$; and $y(1) = 1+1/e$. The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$

The maximum absolute errors are presented in tables 7 for different values of ε.

5. Discussions and conclusions

We have described a fitted Galerkin method for solving a singular perturbation problem with layer behaviour. We have introduced a fitting factor in the Galerkin difference scheme which takes care of the rapid changes that occur in the boundary layer region and its value obtained from the theory of singular perturbations. We have presented maximum absolute errors for the standard examples chosen from the literature and also presented maximum absolute errors for the some of the examples with and without fitting factor to show the efficiency of the method when $\varepsilon << h$. One can extend this method to solve singular-singular perturbation two-point boundary value problem.

Table 1. The maximum absolute errors in solution of example 1 with fitting factor

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	2.85(-2)	2.63(-2)	2.59(-2)	2.59(-2)	2.59(-2)	2.59(-2)	2.59(-2)	2.59(-2)
2^{-4}	1.29(-2)	1.50(-2)	1.39(-2)	1.37(-2)	1.37(-2)	1.37(-2)	1.37(-2)	1.37(-2)
2^{-5}	1.00(-2)	6.80(-3)	7.70(-3)	7.20(-3)	7.10(-3)	7.00(-3)	7.00(-3)	7.00(-3)
2^{-6}	1.50(-2)	5.60(-3)	3.50(-3)	3.90(-3)	3.70(-3)	3.60(-3)	3.60(-3)	3.60(-3)
2^{-10}	2.00(-2)	1.06 (-2)	5.30(-3)	2.50(-3)	1.10(-3)	3.84(-4)	2.23(-4)	2.49(-4)

Table 2. The maximum absolute errors in solution of example 2 with fitting factor

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	1.12(-1)	1.04(-1)	1.02(-1)	1.02(-1)	1.02(-1)	1.02(-1)	1.02(-1)	1.02(-1)
2^{-4}	4.64(-2)	5.94(-2)	5.64(-2)	5.58(-2)	5.56(-2)	5.56(-2)	5.56(-2)	5.56(-2)
2^{-5}	5.97(-2)	2.17(-2)	3.09(-2)	2.96(-2)	2.94(-2)	2.93(-2)	2.93(-2)	2.93(-2)
2^{-6}	8.59(-2)	3.70(-2)	1.10(-2)	1.58(-2)	1.52(-2)	1.51(-2)	1.51(-2)	1.51(-2)
2^{-10}	1.13(-1)	6.85(-2)	3.64(-2)	1.78(-2)	7.80(-3)	2.80(-3)	8.17(-4)	1.00(-3)

Table 3. The maximum absolute errors in solution of example 3 with fitting factor

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	0.11(-15)	0.16(-15)	0.99(-15)	0.77(-15)	0.29(-14)	0.22(-13)	0.67(-13)	0.61(-13)
2^{-4}	0.11(-15)	0.38(-15)	0.33(-15)	0.26(-14)	0.26(-14)	0.15(-13)	0.19(-13)	0.14(-12)
2^{-5}	0.55(-16)	0.22(-15)	0.44(-15)	0.44(-15)	0.16(-14)	0.11(-13)	0.20(-13)	0.19(-12)
2^{-6}	0.11(-15)	0.11(-15)	0.33(-15)	0.24(-14)	0.70(-14)	0.28(-13)	0.11(-12)	0.42(-12)
2^{-10}	0.00(+00)	0.00(+00)	0.00(+00)	0.00(+00)	0.44(-15)	0.11(-15)	0.22(-15)	0.37(-13)

Table 4. The maximum absolute errors in solution of example 3 without fitting factor.

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	0.34(-01)	0.78E-02	0.19E-02	0.48E-03	0.12E-03	0.30E-04	0.75E-05	0.19E-05
2^{-4}	0.13(+00)	0.35E-01	0.79E-02	0.19E-02	0.48E-03	0.12E-03	0.30E-04	0.75E-05
2^{-5}	0.35(+00)	0.14E+00	0.35E-01	0.79E-02	0.19E-02	0.48E-03	0.12E-03	0.30E-04
2^{-6}	0.62(+00)	0.35E+00	0.14E+00	0.35E-01	0.79E-02	0.19E-02	0.48E-03	0.12E-03
2^{-10}	7.90(+00)	2.06(+00)	0.91(+00)	0.77(+00)	0.60(+00)	0.35(+00)	0.13(+00)	0.03(+00)

Table 5. The maximum absolute errors in solution of example 4 with fitting factor

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	1.11(-16)	1.33(-15)	1.33(-15)	1.77(-15)	7.66(-15)	5.12(-14)	2.03(-13)	9.51(-14)
2^{-4}	1.11(-16)	3.33(-16)	2.55(-15)	1.33(-15)	6.99(-15)	1.73(-14)	7.03(-14)	3.37(-13)
2^{-5}	3.33(-16)	2.22(-16)	2.22(-16)	3.66(-15)	4.55(-15)	6.88(-15)	2.77(-14)	1.06(-13)
2^{-6}	6.66(-16)	2.22(-16)	2.22(-16)	3.33(-16)	3.55(-15)	6.72(-14)	1.87(-13)	1.40(-12)
2^{-10}	0.00(+00)	0.00(+00)	0.00(+00)	0.00(+00)	6.66(-16)	2.22(-16)	2.22(-16)	3.33(-16)

Table 6. The maximum absolute errors in solution of example 4 without fitting factor

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	3.44(-2)	7.80(-3)	1.90(-3)	4.77(-4)	1.19(-4)	2.98(-5)	7.45(-6)	1.86(-6)
2^{-4}	1.35(-2)	3.45(-2)	7.90(-3)	1.90(-3)	4.79(-4)	1.19(-4)	2.99(-5)	7.48(-6)
2^{-5}	3.51(-1)	1.35(-1)	3.45(-2)	7.90(-3)	1.90(-3)	4.79(-4)	1.19(-4)	2.99(-5)
2^{-6}	6.27(-1)	3.51(-1)	1.35(-1)	3.45(-2)	7.90(-3)	1.90(-3)	4.79(-4)	1.19(-4)
2^{-10}	7.90(+0)	2.06(+0)	9.17(-1)	7.77(-1)	6.00(-1)	3.51(-1)	1.35(-1)	3.45(-2)

Table 7. The maximum absolute errors in solution of example 5

ε \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	5.32(-2)	4.78(-2)	4.67(-2)	4.69(-2)	4.68(-2)	4.68(-2)	4.68(-2)	4.68(-2)
2^{-4}	2.54(-2)	2.63(-2)	2.39(-2)	2.33(-2)	2.32(-2)	2.32(-2)	2.32(-2)	2.32(-2)
2^{-5}	1.03(-2)	1.29(-2)	1.31(-2)	1.19(-2)	1.16(-2)	1.16(-2)	1.15(-2)	1.15(-2)
2^{-6}	1.54(-2)	5.70(-3)	6.50(-3)	6.50(-3)	5.90(-3)	5.80(-3)	5.80(-3)	5.80(-3)
2^{-10}	2.02(-2)	1.06(-2)	5.30(-3)	2.50(-3)	1.10(-3)	3.84(-4)	4.10(-4)	4.05(-4)

References

[1] Bawa, R. K. and Natesan, S. 2009. An Efficient Hybrid Numerical Scheme for Convection- Dominated Boundary-Value Problems, *International Journal of Computer Mathematics*, 86, 2: 261-273.

[2] Bellman, R. 1964. "*Perturbation Techniques in Mathematics, Physics and Engineering",* Holt, Rinehart, Winston, New York.

[3] Bender, C. M. and Orszag, S. A. 1978. "*Advanced Mathematical Methods for Scientists and Engineers"*, Mc. Graw-Hill, New York.

[4] Eckhaus, W. 1973. "*Matched Asymptotic Expansions and Singular Perturbations",* North Holland Publishing company, Amsterdam, Holland.

[5] Hemker, P. W. and Miller, J. J. H. 1979. "*Numerical Analysis of Singular Perturbation Problems"*, Academic Press, New York.

[6] Kadalbajoo, M. K. and Reddy, Y. N. 1989. asymptotic and Numerical Analysis of singular perturbation problems: A Survey, *Applied Mathematics and computation*, 30: 223-259.

[7] Kadalbajoo, M. K. and Patidar, K. C. 2003. Exponentially Fitted Spline in-Compression for the Numerical Solution of Singular Perturbation Problems, *Computers and Mathematics with Applications*, 46: 751-767.

[8] Kasi Viswanadham, K. N. S., Muralikrishna, P., and Prabhakara Rao, C. 2010. Numerical Solution of Fifth Order Boundary Value Problems by Collocation Method with Sixth Order B-Splines, *International Journal of Applied Science and Engineering,* 8, 2: 119-125.

[9] Kevorkian, J. and Cole, J. D. 1981. "*Perturbation Methods in Applied Mathematics"*, Springer-Verlag, New York.

[10] Natesan, S. and Ramanujam, N. 1999. Improvement of Numerical Solution of Self- Adjoint Singular Perturbation Problems by Incorporation of Asymptotic Approximations, *Applied Mathematics and Computation*, 98: 119-137.
[11] Nayfeh, A. H. 1973. "*Perturbation Methods",* Wiley, New York.
[12] O' Malley, R.E. 1974. "*Introduction to Singular Perturbations"*, Academic Press, New York.
[13] Ramos, H., Vigo-Aguiar, J., Natesan, S., Garcia-Rubio, R and Queiruga M.A. 2010. Numerical Solution of Nonlinear Singularly Perturbed Problems on Non-uniform meshes by using a Non-Standard Algorithm, *Journal of Mathematical Chemistry*, 48, 1: 38-54.
[14] Valanarasu, T. and Ramanujam, N. 2003. An Asymptotic Initial Value Method for Singularly Perturbed Boundary Value Problems for Second Order Ordinary Differential Equations, *Journal of Optimization Theory and Application*, 116: 167-182.
[15] Van Dyke, M. 1974. "*Perturbation Methods in Fluid Mechanics"*, Parabolic Press, Stanford, California.
[16] Vigo-Aguiar, J. and Natesan, S. 2006. An Efficient Numerical Method for Singular Perturbation Problems, *Journal of Computational and Applied Mathematics*, 192: 132-141.