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Numerical Treatment of Singularly Perturbed Two-Point Boundary Value Problems with Mixed Condition Using Differential Quadrature Method

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Abstract: This paper presents the application of Differential Quadrature Method (DQM) for finding the numerical solution of singularly perturbed two point boundary value problems with mixed condition. This method is based on the approximation of the derivatives of the unknown functions involved in the differential equations at the mess point of the solution domain. It is an efficient discretization technique in solving initial and/or boundary value problems accurately using a considerably small number of grid points. To demonstrate the applicability of the method, we have solved several model examples and presented the computational results. The computed results have been compared with the exact/approximate solution to show the accuracy and efficiency of the method.

Keywords: Differential Quadrature Method; Singular perturbation; Ordinary differential equation; Two point boundary value problem; Boundary layer

1. Introduction

Singular perturbation problems containing a small perturbation parameter ε, arise very frequently in many branches of applied mathematics such as , fluid dynamics, quantum mechanics, chemical reactor theory, elasticity, aerodynamics, and the other domain of the great world of fluid motion. A few notable examples are Boundary layer problems, the drift-diffusion equation of semiconductor device modelling, the modelling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto hydrodynamics duct problems at high Hartman numbers, etc. These problems have received a significant amount of attention in past and recent years. It is well known fact that the solution of singular perturbation problems exhibits a multi scale character, that is, there are thin transition layer(s) where the solution varies rapidly, while away from the layers(s) the solution behaves regularly and varies slowly. Therefore, the numerical treatment of singularly perturbed problems presents some major computational difficulties. If we apply the existing classical numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behaviour. Thus, more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems. The survey paper by Kadalbajoo and Reddy [9], gives an erudite outline of the singular perturbation problems and

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their treatment starting from Prandtl's paper [14] on fluid dynamical boundary layers. This survey paper will remain as one of the most readable source on singular perturbation problems.

Natesan and Bawa [11], have considered singularly perturbed reaction diffusion Robin boundary-value problems and devised an almost second-order (up to a logarithmic factor) uniformly convergent scheme, which is a proper combination of the classical finite difference scheme and the cubic spline scheme. The proposed scheme has been applied on a piece-wise uniform Shishkin mess. Andargie and Reddy[1] have presented a numerical integration method for the solution of general singularly perturbed two-point boundary value problems with mixed conditions of left or right end boundary layer. In this method the original second order differential equation has been replaced by an approximate first order differential equation with a small deviating argument and then using the trapezoidal formula, a three term recurrence relationship has been obtained and solved by Thomas algorithm.

For a detailed discussion on singular perturbation problems one may refer to the books and high level monographs: O'Malley [13], Nayfeh [12], Kevorkian and Cole [10], Bender and Orszag [4], Farrell et. al. [8], and Roos et. al. [16].

In this paper, Differential Quadrature Method (DQM) has been applied for solving singularly perturbed two-point boundary value problems with mixed condition. This method is a simple and efficient numerical technique, which approximates the derivative with respect to a coordinate direction at a grid point by a weighted linear sum of all the functional values in that direction. The key to DQM is the determination of weighting coefficients for any order derivative discretization.

To the best of the authors knowledge, the Differential Quadrature Method, where approximation of the derivatives have been based on a polynomial of high degree, has not been implemented for the singularly perturbed two-point boundary value problems with mixed condition.

This paper is organized as follows: Section 2 presents the description of the Differential Quadrature Method, including the formula for finding the weighting coefficients for any order derivative discretization and selection of sampling points. Section 3 presents the basic key procedure to solve differential equation with boundary conditions. Section 4 is devoted to the singularly perturbed two-point boundary value problems with mixed conditions and its solution procedure by DQM in detail. The method of finding computational results at uniform grid points and the presentation of results are given in the Section 5 under the heading Numerical Illustration. In the Sub-section 5.1 and 5.2, we have considered four examples of linear or nonlinear nature with left-end boundary layer and two examples of linear nature with right-end boundary layer, respectively and presented Computational results, show the accuracy and efficiency of the method. The conclusions are presented in section 6. The paper ends with the references.

2. Description of the Differential Quadrature Method

The Differential Quadrature Method(DQM) was introduced by Bellman et al.[2, 3] in the early 1970s and, since then, the technique has been successfully employed in finding the solutions of many problems in applied and physical sciences[5,7,17,18]. This method has been predicted by its proponents as a potential alternative to the conventional solution techniques such as the finite difference and finite elements methods. The basic idea of differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of that variable.

In order to show the mathematical representation of the method, we consider a one dimensional field variable $f(x)$ prescribed in a field domain $a = x_1 \le x \le x_N = b$. Let $f_i = f(x_i)$ be the function values specified in a finite set of N discrete points $x_i(i = 1,2,......,N)$ of the field domain in which the end points x_1 and x_N are included. Next, consider the value of the function derivative $d^r f / dx^r$ at some discrete points x_i, and let it be expressed as a linearly weighted some of the function values.

$$f^{(r)}(x_i) = \frac{d^r f(x_i)}{dx^r} = \sum_{J=1}^{N} A_{ij}^{(r)} f_j, \qquad (i = 1,2,......., N) \tag{1}$$

.

where $A_{ij}^{(r)}$ are the weighting coefficients of the $r^{(th)}$-order derivative of the function associated with points x_i. Equation (1) the quadrature rule for a derivative, is the essential basis of the Differential Quadrature Method. Thus using equation (1) for various order derivatives, one may write a given differential equation at each point of its solution domain and obtain the quadrature analog of the differential equation as a set of algebraic equations in terms of the N function values. These equations may be solved, in conjunction with the quadrature analog of the boundary conditions, to obtain the unknown function values provided that the weighting coefficients are known a priori.

In DQM, it is supposed that the solution of a one–dimensional differential equation is approximated by N-terms high degree polynomial:

$$f(x) = \sum_{k=1}^{N} c_k . x^{k-1} \tag{2}$$

where c_k is a constant.

The weighting coefficients may be determined by some appropriate functional approximations; and the approximate functions are referred to as test functions. The primary requirements for the choices of the test functions are of differentiability and smoothness. That is, the test function of the differential equation must be differentiable at least up to the $n^{(th)}$ derivative (here n is the highest order of the differential equation) and sufficiently smooth to be satisfied the condition of the differentiability. Although there can be many choices of the test functions, a convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions $l_j(x)$, where

$$f(x) = \sum_{j=1}^{N} l_j(x) f_j \tag{3}$$

$l_j(x)$ are the monomials of the $(N-1)^{(th)}$ order polynomials. Note that the number of test functions is equal to the number of the sampling points and for completeness, the number of the sampling points should at least be equal to one plus the order of the highest derivatives. Substituting $l_j(x)$ of equation (3) into equation (1), it may be seen that the weighting coefficients can be easily obtained. The detailed procedures can be found in references (Shu and Rechards [18], Quan and Chang [15]).

2.1. The polynomial test function-based weighting coefficients

#The accuracy of differential quadrature solution depends on the accuracy of the weighting coefficients. To obtain accurate weighting coefficients, Quan and Chang [15] derived explicit formulae of the Lagrangian-interpolation-function-based weighting coefficients for the first and second-order derivatives. Shu and Rechards [18] gave a general relationship for any higher order derivatives. These formulae were obtained by considering the test function in the Lagrangian interpolation process as in eq.(1) and (3).These explicit formulae's merit is that highly accurate weighting coefficients may be determined for any number of arbitrarily spaced sampling points.

Villadsen and Michelsen[19] and Quan and Chang [15] have shown that the weighting coefficients of $r^{(th)}$-order derivatives of the Lagrangian interpolation test functions are:

$$A_{i,j}^{(r)} = \frac{d^r}{dx^r} l_j(x_i) \qquad (i, j = 1,2,.......N) \tag{4}$$

where,

$$l_j(x) = \frac{\phi(x)}{(x - x_j)\phi^{(1)}(x_j)}; \quad \phi(x) = \prod_{m=1}^{N}(x - x_m);$$

$$\phi^{(1)}(x_j) = \frac{d\phi(x_j)}{dx} = \prod_{m=1; m\neq j}^{N}(x_j - x_m)$$

and x_i's are the locations of the grid points. N is the number of sampling points. Note that the eq. (4) is valid as long as linearly independent polynomials are used as a trial functions and, thus, the values of the coefficients are affected only by the distribution of the grid points.

Note that the lagrangian interpolation shape functions $l_j(x)$ have following properties

$$l_j(x_i) = \delta_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \tag{5}$$

Using Eqs. (1), (3), and (4) based on Lagrangian interpolation shape functions, Quan and Chang[15] and Shu and Rechards [18] obtained the following weighting coefficients:

$$A_{i,j}^{(1)} = \frac{dl_j(x_i)}{dx} = \frac{\phi^{(1)}(x_i)}{(x_i - x_j)\phi^{(1)}(x_j)}, \qquad (i, j = 1,2,.......N; \quad i \neq j)$$

$$A_{i,j}^{(r)} = \frac{d^r l_j(x_i)}{dx^r} = r\left(A_{i,i}^{(r-1)} A_{i,j}^{(1)} - \frac{A_{i,j}^{(r-1)}}{(x_i - x_j)}\right), \qquad (i, j = 1,2,......N;\ i \neq j; r \geq 2)$$

$$A_{i,i}^{(r)} = \frac{d^r l_i(x_i)}{dx^r} = -\sum_{j=1; i\neq j}^{N} A_{i,j}^{(r)} \ , (i = 1,2,......N;\ r \geq 1) \tag{6}$$

2.2 Choice of sampling points

A convenient and natural choice for the sampling points is that of the equally spaced points. But the Differential Quadrature solutions usually deliver more accurate results with unequally spaced sampling points. A rational basis for the sampling points is provided by the zeros of the orthogonal polynomials. A well accepted kind of sampling points in the DQM is the so called Gauss-Lobatto-Chebyshev sampling points. For a domain specified by $a \leq x \leq b$ and discretised by a set of unequally spaced points (non-uniform grid), then the coordinate of any point i can be evaluated by:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b - a) \tag{7}$$

3. Application to differential equation

The basic key procedure in the DQM is to approximate the derivatives in a differential equation by equation (1). Substituting the equation (1) into the governing equations and equating both sides of the governing equations, we obtain simultaneous equations which can be solved by use of Gauss elimination or other methods. That is, DQM is composed of the following procedure:

(a) The function to be determined is replaced by a group of function values at a group of selected sampling points. Gauss-Lobatto-Chebyshev sampling points are strongly recommended for numerical stability.
(b) Approximate derivatives in a differential equation by these N unknown function values.
(c) Form a system of linear equations and
(d) Solving the system of linear equation yields the desired unknowns.

The proper implementation of boundary condition is very important for the accurate numerical solution of differential equation. Essential and natural boundary condition can be approximated by DQM. Using the technique in solving differential equation, the governing equations are actually satisfied at each sampling point of the domain, so one has one equation for each point, for each unknown. In the resulting system of algebraic equation from the DQM, each boundary condition replaces the corresponding field equation. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

4. Application to singular perturbation problems

To show the applicability of DQM, we consider the singularly perturbed two point boundary value problems of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x);\ p \le x \le q \tag{8}$$

with

$$d_1 y(p) + d_2 y'(p) = c_5 \tag{9}$$

and

$$d_3 y(q) + d_4 y'(q) = c_6 \tag{10}$$

where ε is a small parameter $0 < \varepsilon \le 1$; $p, q, d_1, d_2, d_3, d_4, c_5, c_6$ are given constants; $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[p, q]$. The values of d_1 and d_2 are not zero simultaneously. Similarly, the values of d_3 and d_4 are not zero at the same time.

For finding the solution of the equation (8) with the boundary conditions (9) and (10) by DQM, we have followed the following procedure/steps:

(i) Discritise the interval $[p, q]$, such that $p = x_1 < x_2 < x_3 < \dots\dots < x_N = q$ where, N is the number of sampling/grid points. Denote $y_i = y(x_i)$ and $f_i = f(x_i)$.

Apply the DQM to approximate the derivatives in the equations (8), (9) and (10), that leads to the following discretized form of the equations:

$$\varepsilon \sum_{j=1}^{N} A_{i,j}^{(2)} y_j + a_i \sum_{j=1}^{N} A_{i,j}^{(1)} y_j + b_i y_i - f_i = 0. \quad i = 1, 2, \dots\dots\dots, N \tag{11}$$

with

$$d_1 y_1 + d_2 \sum_{j=1}^{N} A_{1,j}^{(1)} y_j = c_5 \tag{12}$$

and

$$d_3 y_N + d_4 \sum_{j=1}^{N} A_{N,j}^{(1)} y_j = c_6 \tag{13}$$

Use the equation system (12) and (13) to solve for two unknowns y_1 and y_N, which can be expressed in terms of functional values at the interior points as

$$y_1 = \frac{1}{AXN}\left[c_5\left(d_3 + d_4 A_{N,N}^{(1)}\right) - c_6 d_2 A_{1,N}^{(1)} + \sum_{j=2}^{N-1} AXK1.y_j \right] \tag{14}$$

$$y_N = \frac{1}{AXN}\left[c_6\left(d_1 + d_2 A_{1,1}^{(1)}\right) - c_5 d_4 A_{N,1}^{(1)} + \sum_{j=2}^{N-1} AXKN.y_j \right] \tag{15}$$

where:

$$AXN = \left(d_1 + d_2 A_{1,1}^{(1)}\right)\left(d_3 + d_4 A_{N,N}^{(1)}\right) - d_2 d_4 A_{1,N}^{(1)} A_{N,1}^{(1)}$$

$$AXK1 = d_2 d_4 A_{1,N}^{(1)} A_{N,j}^{(1)} - \left(d_3 + d_4 A_{N,N}^{(1)}\right) d_2 A_{1,j}^{(1)}$$

$$AXKN = d_2 d_4 A_{N,1}^{(1)} A_{1,j}^{(1)} - \left(d_1 + d_2 A_{1,1}^{(1)}\right) d_4 A_{N,j}^{(1)}$$

(iv) Apply the equation (11) at all interior points $x_i, (i = 2,3,............, N-1)$, that leads to a system of $(N-2)$ equations with N unknowns.

(v) Use the expression for y_1 and y_N from equation (14) and (15) in the obtained system of equations from step (iv) to get another system of $(N-2)$ equations with $(N-2)$ unknowns $(y_i, i = 2,3,........., N-1)$.

(vi) Solve the system of equations obtained in step (v).

(vii) Use the obtained values $(y_i, i = 2,3,........., N-1)$ from step (v) in the equation (14) and (15) to get the approximate values of y at the boundary points $x = x_1$ and $x = x_N$ respectively.

We have applied the Gaussian elimination method with partial pivoting and employed the double precision Fortran, to solve the obtained system of linear equations in the step (v), for the unknowns $y_2, y_3,......., y_{N-1}$.

5. Numerical illustrations

To demonstrate the applicability of the **DQM**, we have applied it to six singular perturbation problems of linear or non-linear nature and computed the results for different values of N and ε. These examples have been chosen because they have been widely discussed in literature and because approximate/exact solutions are available for comparison.

Note that the DQM results are given at uniform grids $x_i = ih, (i = 0,1,2,......, K)$, with $h = 0.01$ and $K = 100$, which have been interpolated through the use of natural cubic spline interpolation polynomial. For the derivation of this polynomial, we have used the DQM results $(x_i, y_i), i = 1,2,......, N$, where $y_i, i = 1,2,......, N$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, i = 1,2,........, N$ obtained from (7).

To show the accuracy and efficiency of the method, we have also given the computational results (computed from derived cubic spline interpolation polynomial) in terms of Maximum Absolute Error (M.A.E) for the examples-5.1.1 and 5.1.2, at uniform grids $x_i = ih, (i = 0,1,2,......, K)$ with $K = 100,\ h = 0.01$ and $K = 1000,\ h = 0.001$, for various values of

grid points: N and small parameter: ε. In fact, computational results can be given in terms of the mean absolute error or the mean absolute percentage error or in terms of other types of error.
We have compared the DQM results at uniform grid points with the approximate/exact solution available in literature, for different values of N and ε.

5.1. Examples with left-end boundary layer

To show the applicability of the DQM we have applied it to three linear and one nonlinear singular perturbation problems with left-end boundary layer.

Example 5.1.1. Consider the following linear singular perturbation problem from Dorr et. al. ([6], page 80) and Andargie et. al.[1]:

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \qquad 0 \le x \le 1 \text{ with } -y'(0) = 0 \text{ and } y(1) + \varepsilon y'(1) = 1$$

For this example we have $a(x) = 1, b(x) = -1$ and $f(x) = 0$.

The exact solution is given by: $y(x) = \dfrac{m_2 \exp(m_1 x) - m_1 \exp(m_2 x)}{m_2 (1+\varepsilon m_1) \exp(m_1) - m_1 (1+\varepsilon m_2) \exp(m_2)}$

where $m_1 = \dfrac{\left(-1 + \sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$ and $m_2 = \dfrac{\left(-1 - \sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$

The computational results are presented in Table 5.1.1(a), in terms of Maximum Absolute Error (M.A.E.), for various values of N and ε. The Table 5.1.1(b) shows the comparison with exact and Andargie et. al.[1] solution.

Table 5.1.1(a). Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 0.01$ and $h = 0.001$, for example problem 5.1.1.

$\varepsilon \downarrow$	N =16		N =32		N =64		N =82	
	K =100	K =1000	K =100	K =1000	K =100	K =1000	K =100	K =1000
10^{-1}	.7594E-03	.7601E-03	.1783E-03	.1869E-03	.4837E-04	.4888E-04	.3183E-04	.3260E-04
10^{-2}	.9632E-03	.9632E-03	.2010E-03	.2012E-03	.4852E-04	.4888E-04	.2766E-04	.2879E-04
10^{-3}	.1650E-02	.1650E-02	.2784E-03	.2817E-03	.5096E-04	.5120E-04	.2962E-04	.3052E-04
10^{-4}	.2013E-02	.2013E-02	.3660E-03	.4486E-03	.7957E-04	.8160E-04	.4524E-04	.4524E-04
10^{-5}	.2053E-02	.2053E-02	.3797E-03	.4893E-03	.9161E-04	.1155E-03	.5478E-04	.5585E-04
10^{-6}	.2057E-02	.2057E-02	.3811E-03	.4935E-03	.9292E-04	.1194E-03	.5609E-04	.5949E-04
10^{-7}	.2057E-02	.2057E-02	.3812E-03	.4940E-03	.9310E-04	.1199E-03	.5627E-04	.5984E-04
10^{-8}	.2057E-02	.2057E-02	.3813E-03	.4941E-03	.9310E-04	.1199E-03	.5627E-04	.5996E-04
10^{-9}	.2057E-02	.2057E-02	.3812E-03	.4940E-03	.9310E-04	.1199E-03	.5633E-04	.6002E-04

Table 5.1.1(b). Computational results for example-5.1.1

x	Exact Sol- y(x)	DQM Solution-y(x) $N=35$, $K=100, \varepsilon=10^{-4}$, M.A.E. : .259041800E-03	DQM Solution-y(x) $N=85$, $K=100, \varepsilon=10^{-4}$, M.A.E. : .270605100E-04	Andargie Solution-y(x), $\varepsilon=10^{-4}, h=10^{-2}$, $\delta=0.0008$
.00	.3679162	.3680108	.3679263	.3691142
.02	.3753103	.3751303	.3753080	.3757321
.04	.3828914	.3829003	.3828788	.3833029
.06	.3906255	.3905439	.3906134	.3910365
.08	.3985159	.3985699	.3985087	.3989262
.10	.4065656	.4063769	.4065539	.4069751
.20	.4493200	.4494031	.4493118	.4497223
.30	.4965704	.4965973	.4965470	.4969594
.40	.5487897	.5486997	.5487991	.5491581
.60	.6702798	.6701928	.6702921	.6705798
.80	.8186652	.8187017	.8186579	.8188484
1.00	.9999000	.9998693	.9998858	.9999001

Example 5.1.2: Consider the following singular perturbation problem from Andargie et. al. [1]:
$\varepsilon y''(x)+y'(x)=-1-2x;\qquad x\in[0,1],$ with $-y'(0)=1;$ and $y(1)+\varepsilon\, y'(1)=0.$
For this example we have $a(x)=1, b(x)=0$ and $f(x)=-1-2x.$
The exact solution is given by $y(x)=2-x\,(1+x)+\varepsilon[1-2(\varepsilon[1-\exp(-x/\varepsilon)]-x)]$
The computational results are presented in Table 5.1.2(a), in terms of Maximum Absolute Error (M.A.E.), for various values of N and ε. The Table 5.1.2(b) shows the comparison with exact and Andargie et. al.[1] solution.

Table 5.1.2(a). Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i=ih,(i=0,1,2,......,K)$ with $K=100,\ h=0.01$ and $K=1000,\ h=0.001$, for example problem 5.1.2.

$\varepsilon\downarrow$	N =10		N =20		N =40		N =80	
	K =100	K =1000	K =100	K =1000	K =100	K =1000	K =100	K =1000
10^{-1}	.7478E-02	.7478E-02	.1694E-02	.1694E-02	.4011E-03	.4011E-03	.8523E-04	.8523E-04
10^{-2}	.7467E-02	.7467E-02	.1703E-02	.1703E-02	.4052E-03	.4052E-03	.9882E-04	.9894E-04
10^{-3}	.7527E-02	.7527E-02	.1702E-02	.1702E-02	.4051E-03	.4051E-03	.9871E-04	.9871E-04
10^{-4}	.7537E-02	.7537E-02	.1704E-02	.1704E-02	.4054E-03	.4054E-03	.9871E-04	.9871E-04
10^{-5}	.7538E-02	.7538E-02	.1705E-02	.1705E-02	.4054E-03	.4054E-03	.9871E-04	.9871E-04
10^{-6}	.7539E-02	.7539E-02	.1705E-02	.1705E-02	.4054E-03	.4054E-03	.9871E-04	.9871E-04

Table 5.1.2(b). Computational results for example-5.1.2

x	Exact Sol.- y(x)	DQ Solution- y(x) $N=64$, $K=100$, $\varepsilon=10^{-4}$, M.A.E. : 0.1555681E-03	DQ Solution- y(x) $N=128, K=100$, $\varepsilon=10^{-4}$, M.A.E.: 0.3826618E-04	Andargie Solution- y(x), $\varepsilon=10^{-4}, h=10^{-2}$, $\delta=0.0008$
.00	2.0001000	2.0001000	2.0001000	1.9978049
.02	1.9797040	1.9796940	1.9797010	1.9774493
.04	1.9585080	1.9585010	1.9585030	1.9562993
.06	1.9365120	1.9365030	1.9365120	1.9343493
.08	1.9137160	1.9136700	1.9137090	1.9115993
.10	1.8901200	1.8901010	1.8901190	1.8880492
.20	1.7601400	1.7600450	1.7601160	1.7582994
.30	1.6101600	1.6100620	1.6101450	1.6085494
.40	1.4401800	1.4400320	1.4401460	1.4387994
.60	1.0402200	1.0400720	1.0401860	1.0392994
.80	.5602601	.5601645	.5602357	.5597996
1.00	.0003000	.0002999	.0003000	.0002997

Example 5.1.3: Consider the following non-homogeneous singular perturbation problem:

$$\varepsilon\, y''(x) + y'(x) - y(x) = -(1+2x);\ x \in [0,1],\ \text{ with }\ y(0) - y'(0) = 1;\ \text{ and }\ y(1) = 0.$$

For this example we have $a(x) = 1, b(x) = -1$ and $f(x) = -(1+2x)$.

The exact solution is given by: $y(x) = (3+2x) + \dfrac{5(m_2 - 1)\exp(m_1 x) + 5(1-m_1)\exp(m_2 x)}{(1-m_2)\exp(m_1) - (1-m_1)\exp(m_2)}$

where $m_1 = \dfrac{\left(-1+\sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$ and $m_2 = \dfrac{\left(-1-\sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$

The computational results are presented in Table 5.1.3(a) and 5.1.3(b), for different values of N and ε.

Example 5.1.4: Consider the following non-linear singular perturbation problem from Dorr et.al.([6], page 80) and Andargie et. al.[1]:

$$\varepsilon\, y''(x) + y'(x) - y^2(x) = 0;\ x \in [0,1],\ \text{ with }\ -y'(0) = 0\ \text{ and }\ y(1) + \varepsilon\, y'(1) = 1.$$

The linear problem (using quasilinearisation process) concerned to this is :

$$\varepsilon\, y''(x) + \frac{1}{\varepsilon} y'(x) - \frac{2}{\varepsilon\,(x+c)} y(x) = \frac{-1}{\varepsilon\,(x+c)^2};\ x \in [0,1],\ \text{ where, }\ c = \frac{-3-\sqrt{(1+4\varepsilon)}}{2}.$$

For this example we have $a(x) = 1/\varepsilon,\ b(x) = \dfrac{2}{\varepsilon\,(x+c)}$ and $f(x) = \dfrac{-1}{\varepsilon\,(x+c)^2}$.

The asymptotic solution is given by: $y(x) = \dfrac{1}{2-x} + \dfrac{5\exp(-x/\varepsilon)}{4} + O(\varepsilon)$

The computational results are presented in Table 5.1.4(a) and 5.1.4(b), for different values of N and ε.

Table 5.1.3(a). Computational results for example-5.1.3

x	Exact Sol.- y(x)	DQ Solution- y(x) $N=42,\ K=100,\ \varepsilon=10^{-4}$, M.A.E. : 0.5865097E-03	DQ Solution- y(x) $N=128,\ K=100,\ \varepsilon=10^{-4}$, M.A.E. : 0.6133318E-04
.00	1.1604190	1.1604180	1.1604180
.01	1.1619330	1.1619200	1.1619320
.02	1.1632610	1.1632390	1.1632580
.03	1.1644010	1.1643610	1.1644000
.04	1.1653520	1.1653090	1.1653470
.05	1.1661110	1.1660860	1.1661060
.06	1.1666770	1.1665970	1.1666770
.07	1.1670480	1.1670440	1.1670390
.08	1.1672220	1.1671140	1.1672150
.09	1.1671960	1.1671750	1.1671840
.10	1.1669690	1.1668390	1.1669680
.20	1.1531750	1.1530760	1.1531480
.30	1.1169000	1.1167240	1.1168810
.40	1.0557770	1.0555650	1.0557310
.60	.8482658	.8480017	.8482091
.80	.5062645	.5060861	.5062144
1.00	.0000000	.0000000	.0000000

Table 5.1.3(b). Computational results for example-5.1.3

x	Exact Sol.- y(x)	DQ Solution- y(x) $N=62,\ K=100,\ \varepsilon=10^{-5}$, M.A.E. : 0.2585053E-03	DQ Solution- y(x) $N=116,\ K=100,\ \varepsilon=10^{-5}$, M.A.E. : 0.7265806E-04
.00	1.1605840	1.1605850	1.1605850
.01	1.1620980	1.1620960	1.1620970
.02	1.1634260	1.1634140	1.1634230
.03	1.1645660	1.1645540	1.1645620
.04	1.1655170	1.1655040	1.1655120
.05	1.1662760	1.1662550	1.1662680
.06	1.1668420	1.1668080	1.1668380
.07	1.1672130	1.1671720	1.1672020
.08	1.1673860	1.1673630	1.1673860
.09	1.1673610	1.1673310	1.1673480
.10	1.1671330	1.1670730	1.1671170
.20	1.1533370	1.1533350	1.1533300
.30	1.1170560	1.1168840	1.1170080
.40	1.0559250	1.0557150	1.0558980
.60	.8483864	.8481283	.8483526
.80	.5063381	.5063337	.5063254
1.00	.0000000	.0000000	.0000000

Table 5.1.4(a). Computational results for example-5.1.4

x	Exact Sol.- y(x)	DQ Solution- y(x) $N=42$, $K=100$, $\varepsilon=10^{-4}$, M.A.E. : 0.1478195E-03	DQ Solution- y(x) $N=64$, $K=100$, $\varepsilon=10^{-4}$, M. A. E. : 0.7557869E-04	Andargie Solution- y(x), $\varepsilon=10^{-4}, h=10^{-2}$, $\delta=0.0008$
.00	.5000250	.5000733	.5000203	.4999691
.02	.5050505	.5050783	.5050513	.5050189
.04	.5102041	.5101702	.5102203	.5101719
.06	.5154639	.5154828	.5154799	.5154312
.08	.5208333	.5208430	.5208271	.5207999
.10	.5263158	.5263503	.5262809	.5262817
.20	.5555555	.5556289	.5555524	.5555178
.30	.5882353	.5882078	.5882239	.5881934
.40	.6250000	.6250905	.6250163	.6249532
.60	.7142857	.7142872	.7143192	.7142271
.80	.8333333	.8333221	.8333632	.8332582
1.00	1.0000000	.9999242	.9999244	.9999000

Table 5.1.4(b). Computational results for example-5.1.4

x	Exact Sol.- y(x)	DQ Solution- y(x) $N=36$, $K=100$, $\varepsilon=10^{-5}$, M.A.E. : 0.2641082E-03	DQ Solution- y(x) $N=92$, $K=100$, $\varepsilon=10^{-5}$, M.A.E. : 0.3415346E-04
.00	.5000025	.5001246	.5000170
.01	.5025126	.5025992	.5025234
.02	.5050505	.5050036	.5050616
.03	.5076142	.5077147	.5076291
.04	.5102041	.5102459	.5102120
.05	.5128205	.5127482	.5128098
.06	.5154639	.5155020	.5154731
.07	.5181347	.5182557	.5181395
.08	.5208333	.5209039	.5208350
.09	.5235602	.5235434	.5235698
.10	.5263158	.5262873	.5263174
.20	.5555555	.5556691	.5555509
.30	.5882353	.5882084	.5882525
.40	.6250000	.6250774	.6250234
.60	.7142857	.7145137	.7143076
.80	.8333333	.8335262	.8333624
1.00	1.0000000	.9999924	.9999924

5.2. Examples with right-end boundary layer

To demonstrate the applicability of the DQM, we have applied it to two linear singular perturbation problems with right-end boundary layer.

Example5.2.1. Consider the following non-homogeneous singular perturbation problem from Andargie et. al.[1]: $-\varepsilon y''(x)+y'(x)=3-2x;$ $0\le x\le 1$ **,**with $y(0)-\varepsilon y'(0)=0$ and $y'(1)=1$.

For this example we have $a(x) = -1, b(x) = 0$ and $f(x) = 2x - 3$. The exact solution is given by: $y(x) = x(3 - x - 2\varepsilon) + \varepsilon[3 - 2\varepsilon(1 - \exp((x-1)/\varepsilon))]$

The computational results are presented in Table 5.2.1(a) and 5.2.1(b), for different values of N and ε.

Table 5.2.1(a). Computational results for example-5.2.1

x	Exact Sol.- y(x)	DQ Solution- y(x) $N = 52$, $K = 100$, $\varepsilon = 10^{-4}$, M.A.E. : 0.2371073E-03	DQ Solution- y(x) $N = 116$, $K = 100$, $\varepsilon = 10^{-4}$, M.A.E. : 0.4673004E-04	Andargie Solution- y(x), $\varepsilon = 10^{-4}, h = 10^{-2}$, $\delta = 0.0008$
.00	.0003000	.0003000	.0002999	.0002953
.20	.5602599	.5602287	.5602539	.5596657
.40	1.0402200	1.0400580	1.0402000	1.0391645
.60	1.4401800	1.4400180	1.4401600	1.4386631
.80	1.7601400	1.7601090	1.7601340	1.7581611
.90	1.8901200	1.8900350	1.8901030	1.8879100
.92	1.9137160	1.9136550	1.9137150	1.9114598
.94	1.9365120	1.9365040	1.9365070	1.9342095
.96	1.9585080	1.9584720	1.9585020	1.9561592
.98	1.9797040	1.9796860	1.9797000	1.9773091
1.00	2.0001000	2.0001000	2.0000990	1.9976645

Table 5.2.1(b). Computational results for example-5.2.1

x	Exact Sol,- y(x)	DQ Solution- y(x) $N = 42$, $K = 100$, $\varepsilon = 10^{-5}$, M.A.E. : 0.3668070E-03	DQ Solution- y(x) $N = 116$, $K = 100$, $\varepsilon = 10^{-5}$, M.A.E. : 0.4673004E-04
.00	.0000300	.0000300	.0000300
.20	.5600260	.5599381	.5600199
.40	1.0400220	1.0398660	1.0400020
.60	1.4400180	1.4398620	1.4399980
.80	1.7600140	1.7599260	1.7600070
.90	1.8900120	1.8898840	1.8899950
.91	1.9019120	1.9018920	1.9018980
.92	1.9136120	1.9135040	1.9136110
.93	1.9251110	1.9251080	1.9250990
.94	1.9364110	1.9363290	1.9364060
.95	1.9475110	1.9474850	1.9475020
.96	1.9584110	1.9583660	1.9584050
.97	1.9691110	1.9690690	1.9691060
.98	1.9796100	1.9795880	1.9796060
.99	1.9899100	1.9898970	1.9899080
1.00	2.0000100	2.0000100	2.0000090

Example 5.2.2. Consider the following singular perturbation problem from Dorr et. al.([[6], page 80] with $a = 1, n = 1$) and Andargie et. al.[1]:

$\varepsilon y''(x) - y'(x) - y(x) = 0;$ $\quad 0 \le x \le 1$ with $y(0) - y'(0) = 1$ and $y'(1) = 0.$

For this example we have $a(x) = -1, b(x) = -1$ and $f(x) = 0.$

The exact solution is given by: $y(x)=\dfrac{m_1\exp(m_2 x)-m_2\exp(m_1(x-1)+m_2)}{m_1(1-\varepsilon\, m_2)-m_2(1-\varepsilon\, m_1)\exp(m_2-m_1)}$ where $m_1=\dfrac{\left(1+\sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$ and $m_2=\dfrac{\left(1-\sqrt{(1+4\varepsilon)}\right)}{2\varepsilon}$.

The computational results are presented in Table 5.2.2(a) and 5.2.2(b), for different values of N and ε.

Table 5.2.2(a). Computational results for example-5.2.2

x	Exact Sol-y(x)	DQ Solution-y(x) $N=44$, $K=100$, $\varepsilon=10^{-4}$, M.A.E. : 0.1854897E-03	DQ Solution-y(x) $N=98$, $K=100$, $\varepsilon=10^{-4}$, M.A.E. : 0.2855062E-04	Andargie solution-y(x), $\varepsilon=10^{-4}, h=10^{-2}$ $\delta=0.0008$
.00	.9999000 .8186653 .6702799 .5487897	.9999282	.9999106	.9999009
.20	.4493200	.8188417	.8186924	.8188949
.40	.4065656	.6704614	.6702996	.6706179
.60	.3985159	.5488610	.5488139	.5491892
.80	.3906255	.4493664	.4493309	.4497477
.90	.3828914	.4065627	.4065794	.4069981
.92	.3753104	.3986197	.3985273	.3989487
.94	.3679162	.3906230	.3906274	.3910585
.96		.3829324	.3828988	.3833245
.98		.3754094	.3753099	.3757532
1.00		.3680357	.3679301	.3691349

Table 5.2.2(b). Computational results for example-5.2.2

x	Exact Sol-y(x)	DQ Solution- y(x) $N=56$, $K=100$, $\varepsilon=10^{-5}$, M.A.E : 0 .1195669E-03	DQ Solution- y(x) $N=86$, $K=100$, $\varepsilon=10^{-5}$, M.A.E : 0 .4959106E-04
.00	.9999900	.9999935	.9999933
.20	.8187242	.8187792	.8187711
.40	.6703160	.6703058	.6703593
.60	.5488095	.5489030	.5488055
.80	.4493281	.4494078	.4493231
.90	.4065693	.4065762	.4065806
.91	.4025239	.4025403	.4025402
.92	.3985187	.3985990	.3985339
.93	.3945534	.3945704	.3945684
.94	.3906276	.3906401	.3906351
.95	.3867408	.3868126	.3867671
.96	.3828928	.3828625	.3828840
.97	.3790829	.3791558	.3790896
.98	.3753110	.3752779	.3753292
.99	.3715767	.3715966	.3715815
1.00	.3678831	.3679643	.3679164

6. Discussion and conclusions

In this paper, the DQM has been applied to solve three linear and one non-linear singular perturbation problems with left-end boundary layer and, two linear singular perturbation problem with right-end boundary layer. The applications presented here showed that the DQM has the capability of solving general singularly perturbed two point boundary value problems with mixed condition and of producing accurate solutions with minimal computational effort. It can be observed from the tables that the DQM approximates the exact or asymptotic or approximate solution very well with small number of sampling points. This shows the efficiency and accuracy of the present method. We have given here only a few values although the solutions can be computed at desired number of uniform points.

It has been observed that an increase in the number of grid points gives rise to an increase in the accuracy of the DQM solution, as in the most numerical techniques. However a small number of grid points in the DQM produces highly accurate results with the use of non-uniform grids. This method provides an alternative technique to the conventional ways of solving singular perturbation problems.

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