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An Improved Algorithm for Solving Fuzzy Maximal Flow Problems

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Abstract: Kumar et al. (A new approach for solving fuzzy maximal flow problems; Lecture Notes in Computer Science, Springer-Verlag; Berlin Heidelberg, 5908, (2009) 278-286) proposed a new algorithm for solving the fuzzy maximal flow problems. In the numerical example solved by Kumar et al., by using the proposed algorithm, there exist negative part in the obtained triangular fuzzy numbers which represents that the flow between two nodes may be negative. Although it is mathematically correct according to properties of ranking function. But in real life problems the negative quantity of the flow has no physical meaning. To overcome this shortcoming the algorithm, proposed by Kumar et al., is improved.

Keywords: Fuzzy maximal flow problem; ranking function; triangular fuzzy number.

1. Introduction

The maximal flow problem is one of basic problems for combinatorial optimization in weighted directed graphs. It provides very useful models in a number of practical contexts including communication networks, oil pipeline systems and power systems. The maximal flow problem and its variations have wide range of applications and have been studied extensively. The maximal flow problem was proposed by Fulkerson and Dantzig (1955) [3] originally and solved by specializing the simplex method for the linear programming, and Ford and Fulkerson (1956) [8] solved it by augmenting path algorithm. There are efficient algorithms to solve the crisp maximal flow problems (Ahuja et al. (1993) [1], Bazarra et al. (1990)) [2].

In the real life situations there always exist uncertainty about the parameters (e.g.: costs, capacities and demands) of maximal flow problems. To deal with such type of problems, the parameters of maximal flow problems are represented by fuzzy numbers (Zadeh (1965)) [8] and maximal flow problems with fuzzy parameters are known as fuzzy maximal flow problems. In the literature, the numbers of papers dealing with fuzzy maximal flow problems are less (Chanas and Kolodziejczyk (1982, 1984, 1986)) [4-6]. The paper by Kim and Roush (1982) [13] is one of the first on this subject. The authors developed the fuzzy flow theory, presenting the conditions to obtain a optimal flow, by means of definitions on fuzzy matrices. But there were Chanas and Kolodziejczyk (1982,1984,1986) [4-6] who introduced the main works in the literature involving this subject. They approached this problem using the minimum cuts technique.

In the first paper, Chanas and Kolodziejczyk (1982) [4] presented an algorithm for

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a graph with crisp structure and fuzzy capacities, i.e., the arcs have a membership function associated in their flow. This problem was studied by Chanas and Kolodzijczyk (1984) [5] again, in this paper the flow is a real number and the capacities have upper and lower bounds with a satisfaction function. Chanas and Kolodzijczyk (1986) [6] had also studied the integer flow and proposed an algorithm. Chanas et al. (1995) [3] studied the maximum flow problem when the underlying associated structure is not well defined and must be modeled as a fuzzy graph. Diamond (2001) [7] developed interval-valued versions of the max-flow min cut theorem and Karp-Edmonds algorithm and provide robustness estimates for flows in networks in an imprecise or uncertain environment. These results are extended to networks with fuzzy capacities and flows.

Liu and Kao (2004) [16] investigated the network flow problems in that the arc lengths of the network are fuzzy numbers. Ji et al. (2006) [11] considered a generalized fuzzy version of maximum flow problem, in which arc capacities are fuzzy variables. Hernandes et al. (2007) [10] proposed an algorithm, based on the classic algorithm of Ford-Fulkerson. The algorithm uses the technique of the incremental graph and representing all the parameters as fuzzy numbers. Kumar et al. (2009) [14] proposed a new algorithm to find fuzzy maximal flow between source and sink by using ranking function.

In this paper the shortcomings of the existing algorithms (Kumar et al. (2009)) [14] are pointed out and to overcome these shortcomings an improved algorithm is introduced for solving the fuzzy maximal flow problems.

This paper is organized as follows: In Section 2, some basic definitions and ranking function are reviewed. In Section 3, the shortcomings of the existing algorithm (Kumar et al. (2009)) [14] are pointed out. In Section 4, the arithmetic operations of triangular fuzzy numbers are reviewed. In Section 5, an improved algorithm is proposed for solving the fuzzy maximal flow problems. In Section 6, to illustrate the improved algorithm and to point out the shortcomings of the existing algorithm (Kumar et al. (2009)) [14] a numerical example is solved. The obtained results are discussed in Section 7. The conclusions are discussed in Section 8.

2. Preliminaries

In this section, some basic definitions and ranking function are presented.

2.1. Basic Definitions

In this subsection, some basic definitions are presented (Kaufmann and Gupta (1985)).

Definition 1: The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range $[0,1]$ i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned values indicate the membership grade of the element in the set A.

The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set.

Definition 2: A fuzzy number $\tilde{A}$ is said to be a non-negative fuzzy number if and only if $\mu_{\tilde{A}}(x) = 0, \forall\, x < 0$.

Definition 3: A fuzzy number $\tilde{A} = (a,b,c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & a \le x \le b \\ 1 & x = b \\ \dfrac{(x-c)}{(b-c)}, & b \le x \le c \end{cases}$$

where $a,b,c \in R$

Definition 4: A triangular fuzzy number $\widetilde{A}=(a,b,c)$ is said to be non-negative triangular fuzzy number if and only if $a \geq 0$.

Definition 5: A triangular fuzzy number $\widetilde{A}=(a,b,c)$ is said to be zero triangular fuzzy number if and only if $a=0, b=0, c=0$.

2.2. Ranking Function

An efficient approach for comparing the fuzzy numbers is by the use of ranking function (Liou and Wang (1992)). A ranking function $\Re$: F(R) $\rightarrow$ R, where F(R) is set of all fuzzy numbers defined on set of real numbers, which maps each fuzzy number into a real number.

Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers, then (i) $\widetilde{A} \succ \widetilde{B}$ if $\Re(\widetilde{A}) > \Re(\widetilde{B})$ (ii) $\widetilde{A} \prec \widetilde{B}$ if $\Re(\widetilde{A}) < \Re(\widetilde{B})$ (iii) $\widetilde{A} \approx \widetilde{B}$ if $\Re(\widetilde{A}) = \Re(\widetilde{B})$

For a triangular fuzzy number $\widetilde{A}=(a,b,c)$ ranking function $\Re$ is given by $\Re(\widetilde{A}) = \frac{1}{4}(a+2b+c)$.

3. Shortcomings of the existing algorithm

Kumar et al. (2009) [14] proposed a new algorithm to find the fuzzy maximal flow between source and sink by using ranking function and representing the flow between nodes as triangular fuzzy numbers. In the numerical example solved by Kumar et al. (2009), by using the proposed algorithm, there exist negative part in the obtained triangular fuzzy numbers which represents that the flow between two nodes may be negative. Although it is mathematically correct according to properties of ranking function. But in real life problems the negative quantity of the flow has no physical meaning.

For example in the optimal network Kumar et al. (2009) the flow from node 1 to node 3, from node 2 to node 5 and from node 5 to node 2 are triangular fuzzy numbers $(-50,0,50)$, $(-35,10,55)$ and $(-20,20,60)$ respectively. Since $\Re(-50,0,50)$ is zero and $\Re(-35,10,55), \Re(-20,20,60)$ are positive so according to properties of ranking functions these flows are non-negative fuzzy numbers. But there exist negative part in all the obtained triangular fuzzy numbers, which represents that the flow between these nodes may be negative. But the negative quantity of flow has no physical meaning.

4. Arithmetic operations of triangular fuzzy numbers

In the existing algorithm (Kumar et al. (2009)) the following arithmetic operations are used:

Let $\widetilde{A}=(a_1,b_1,c_1)$ and $\widetilde{B}=(a_2,b_2,c_2)$ be two triangular fuzzy numbers then

(i) $\widetilde{A} \oplus \widetilde{B} = (a_1+a_2,\ b_1+b_2,\ c_1+c_2)$

(ii) $\widetilde{A} \ominus \widetilde{B} = (a_1-c_2,\ b_1-b_2,\ c_1-a_2)$

(iii) $$\text{minimum}(\widetilde{A},\widetilde{B}) = \begin{cases} \widetilde{A}, & \text{if } \Re(\widetilde{A}) < \Re(\widetilde{B}) \\ \widetilde{B}, & \text{if } \Re(\widetilde{A}) > \Re(\widetilde{B}) \\ \widetilde{A} \text{ or } \widetilde{B}, & \text{if } \Re(\widetilde{A}) = \Re(\widetilde{B}) \end{cases}$$

To overcome the shortcomings, pointed out in Section 3, the subtraction operation, used in existing algorithm (Kumar et al. (2009)), is replaced by the following operation (Soltani and Haji (2007)):

Let $\widetilde{A}=(a_1,b_1,c_1)$ and $\widetilde{B}=(a_2,b_2,c_2)$ be two triangular fuzzy numbers then

$\widetilde{A} \ominus \widetilde{B} = (a,b,c)$,

where,

$$\begin{cases} c = \text{maximum}(0,(c_1-c_2)) \\ b = \text{maximum}(0,\text{minimum}(c,(b_1-b_2))) \\ a = \text{maximum}(0,\text{minimum}(b,(a_1-a_2))) \end{cases}$$

5. Improved algorithm

In this section, the existing algorithm (Kumar et al. (2009)) [14] is improved for solving the fuzzy maximal flow problems.

The improved algorithm is a labeling technique. Since improved algorithm is direct extension of existing algorithm (Taha (2003)) [18], so it is very easy to understand and apply for solving fuzzy maximal problems occuring in real life situations.

The fuzzy maximal flow algorithm is based on finding breakthrough paths with net positive flow between the source and sink nodes. Consider arc (i, j) with initial fuzzy capacities $(\widetilde{f}\,\overline{c}_{ij}, \widetilde{f}\,\overline{c}_{ji})$. As portions of these fuzzy capacities are committed to the flow in the arc, the fuzzy residuals (or remaining fuzzy capacities) of the arc are updated. We use the notation $(\widetilde{fc}_{ij}, \widetilde{fc}_{ji})$ to represent these fuzzy residuals.

For a node j that receives flow from node i, we attach a label $[\widetilde{fa}_j, i]$, where $\widetilde{fa}_j$ is the fuzzy flow from node i to j. The steps of the algorithm are thus summarized as follows:

Step 1. For all arcs (i, j), set the residual fuzzy capacity equal to the initial fuzzy capacity i.e., $(\widetilde{fc}_{ij}, \widetilde{fc}_{ji}) = (\widetilde{f}\,\overline{c}_{ij}, \widetilde{f}\,\overline{c}_{ji})$. Let $\widetilde{fa}_1 = \widetilde{\infty}$ and label source 1 with $[\widetilde{\infty}, -]$. Set $i = 1$, and go to step 2.

Step 2. Determine S_i, the set of unlabeled nodes j that can be reached directly from node i by arcs with positive residuals (i.e., $\widetilde{fc}_{ij}$ is a non-negative fuzzy number for all $j \in S_i$). If $S_i \neq \phi$, go to step 3. Otherwise, go to step 4.

Step 3. Determine $k \in S_i$ such that

$$\underset{j \in S_i}{\text{maximum}}\{\Re(\widetilde{fc}_{ij})\} = \Re(\widetilde{fc}_{ik})$$

Set $\widetilde{fa}_k = \widetilde{fc}_{ik}$ and label node k with $[\widetilde{fa}_k, i]$. If $k = n$, the sink node has been labeled, and a breakthrough path is found, go to step 5. Otherwise, set $i = k$, and go to step 2.

Step 4. (Backtracking). If $i = 1$, no breakthrough is possible; go to step 6. Otherwise, let r be the node that has been labeled immediately before current node i and remove i from the set of nodes adjacent to r. Set $i = r$, and go to step 2.

Step 5. (Determination of residuals). Let $N_p = (1, k_1, k_2, \ldots, n)$ define the nodes of the p^{th} breakthrough path from source node 1 to sink node n. Then the maximal flow along the path is computed as $\widetilde{f}_p = \text{minimum}\{\widetilde{fa}_1, \widetilde{fa}_{k_1}, \widetilde{fa}_{k_2}, \ldots, \widetilde{fa}_n\}$

The residual capacity of each arc along the breakthrough path is decreased by $\widetilde{f}_p$ in the direction of the flow and increased by $\widetilde{f}_p$ in the reverse direction i.e., for nodes i and j on the path, the residual flow is changed from the current $(\widetilde{fc}_{ij}, \widetilde{fc}_{ji})$ to

(a) $(\widetilde{fc}_{ij} \ominus \widetilde{f}_p, \widetilde{fc}_{ji} \oplus \widetilde{f}_p)$ if the flow is from i to j

(b) $(\widetilde{fc}_{ij} \oplus \widetilde{f}_p, \widetilde{fc}_{ji} \ominus \widetilde{f}_p)$ if the flow is from j to i

Reinstate any nodes that were removed in step 4. Set $i = 1$, and return to step 2 to attempt a new breakthrough path.

Step 6. (Solution). Given that m breakthrough paths have been determined, the fuzzy maximal flow in the network is $\widetilde{F} = \widetilde{f}_1 \oplus \widetilde{f}_2 \oplus \ldots \oplus \widetilde{f}_m$ where m is the number of iteration to get no breakthrough.

6. Illustrative example

In this section the improved algorithm is illustrated by solving a numerical example.
Example 1**.** Consider the network shown in Figure 1. The bidirectional fuzzy capacities are shown on the respective arcs. For example, for arc $(3,4)$ the flow limit is approximately 10 say $(5,10,15)$ units from node 3 to 4 and approximately 5 say $(0,5,10)$ units from node 4 to 3. Determine the fuzzy maximal flow in this network between source 1 and sink 5 .

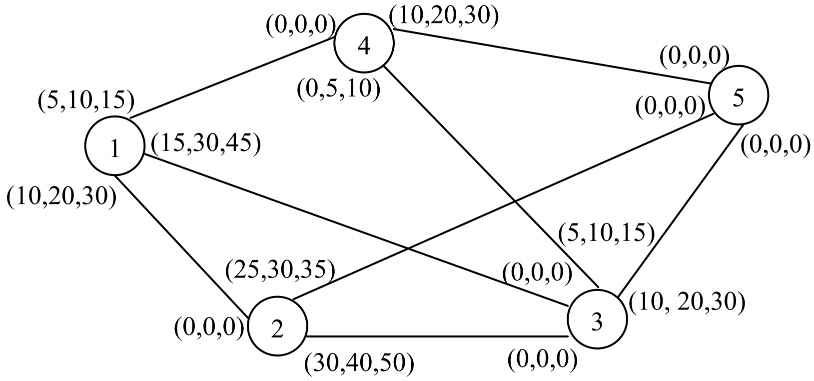

Figure 1. A network for the maximal flow problem with fuzzy arc lengths

The algorithm is applied in the following manner:

Iteration 1. Set the initial residuals $(\widetilde{fc}_{ij}, \widetilde{fc}_{ji})$ equal to the initial capacities $(\widetilde{f}\ \bar{c}_{ij}, \widetilde{f}\ \bar{c}_{ji})$.

Step 1. Set $\widetilde{fa}_1 = (\infty,\infty,\infty)$ and label node 1 with $[(\infty,\infty,\infty),-]$. Set $i = 1$.

Step 2. $S_1 = \{2,3,4\}\ (\neq \phi)$.

Step 3. $k = 3,$ because
$\text{maximum}\{\Re(\widetilde{fc}_{12}),\ \Re(\widetilde{fc}_{13}),\ \Re(\widetilde{fc}_{14})\} = \Re(\widetilde{fc}_{13})$
i.e., $\text{maximum}\{20, 30, 10\} = 30$. Set $\widetilde{fa}_3 = \widetilde{fc}_{13} = (15, 30, 45),$ and label node 3 with $[(15, 30, 45), 1]$. Set $i = 3,$ and repeat step 2.

Step 4. $S_3 = \{4,5\}$.

Step 5. $k = 5$, because
$\text{maximum}\{\Re(\widetilde{fc}_{34}),\ \Re(\widetilde{fc}_{35})\} = \Re(\widetilde{fc}_{35})$. Set $\widetilde{fa}_5 = \widetilde{fc}_{35} = (10, 20, 30)$, and label node 5 with $[(10, 20, 30)], 3]$. Breakthrough is achieved. Go to step 5.

Step 6. The breakthrough path is determined from the labels starting at node 5 and moving backward to node 1 and the breakthrough path is $1 \to 3 \to 5$. Thus $N_1 = \{1, 3, 5\}$ and
$\widetilde{f}_1 = \text{minimum}\{\widetilde{fa}_1, \widetilde{fa}_3, \widetilde{fa}_5\} = \text{minimum}\{(\infty,\infty,\infty),$ $(15, 30, 45),\ (10, 20, 30)\} = (10, 20, 30)$. The residual capacities along path N_1 are

$$(\widetilde{fc}_{13}, \widetilde{fc}_{31}) = \big((15,30,45)\ \Theta\ (10,20,30), (0,0,0) \oplus (10,20,30)\big) = \big((5,10,15), (10,20,30)\big)$$

$$(\widetilde{fc}_{35}, \widetilde{fc}_{53}) = \big((10,20,30)\ \Theta\ (10,20,30), (0,0,0) \oplus (10,20,30)\big) = \big((0,0,0), (10,20,30)\big)$$

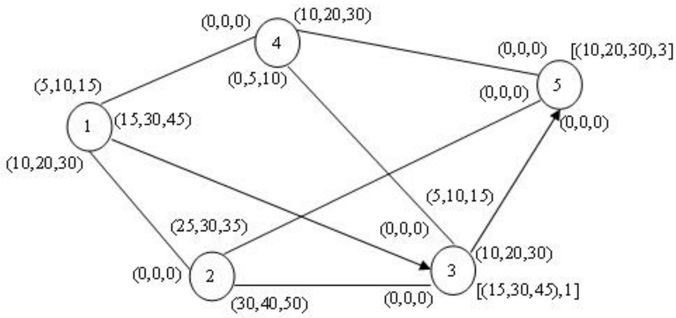

Figure 2. Obtained network after iteration 1

Iteration 2. Repeating the procedure described in the first iteration, at the starting node 1, the obtained breakthrough path is $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$ and $\widetilde{f}_2 = (5,10,15)$.

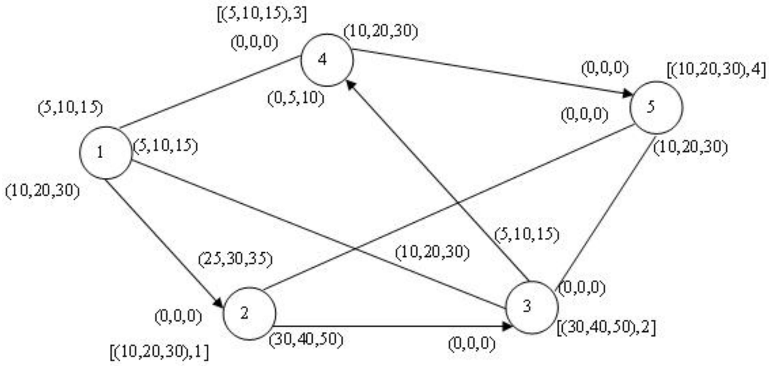

Figure 3. Obtained network after iteration 2

Iteration 3. The obtained breakthrough path is $1 \rightarrow 2 \rightarrow 5$ and $\widetilde{f}_3 = (5,10,15)$.

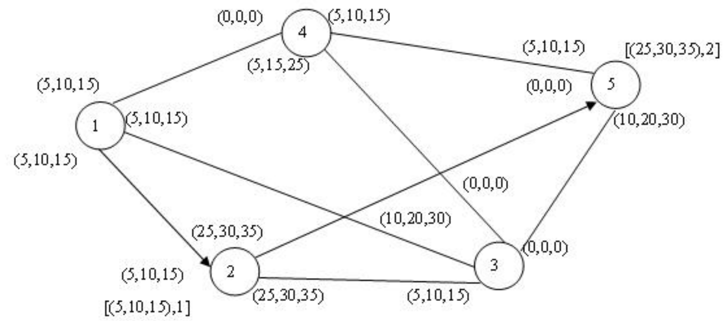

Figure 4. Obtained network after iteration 3

Iteration 5. The obtained breakthrough path is $1 \rightarrow 4 \rightarrow 5$ and $\widetilde{f}_5 = (5,10,15)$.

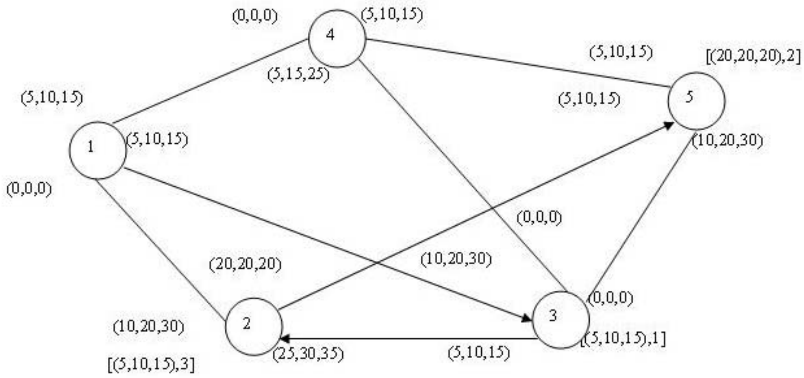

Figure 5. Obtained network after iteration 4

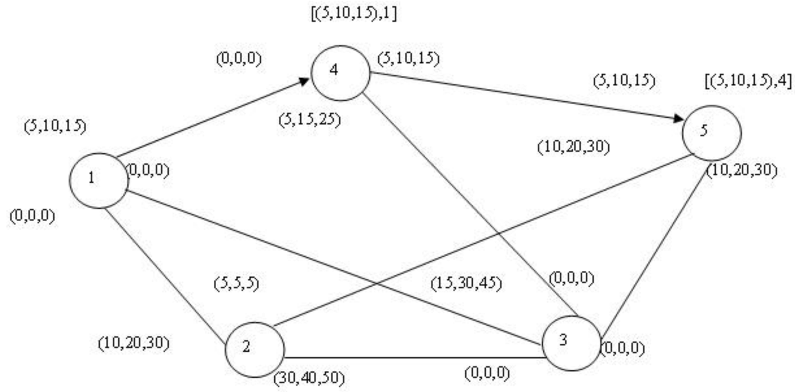

Figure 6. Obtained network after iteration 5

More iterations are not possible after 5th iteration because there is no way out to reach at sink from source. The fuzzy maximal flow is:

$$\widetilde{F} = \widetilde{f}_1 \oplus \widetilde{f}_2 \oplus \widetilde{f}_3 \oplus \widetilde{f}_4 \oplus \widetilde{f}_5 = (10,20,30) \oplus (5,10,15) \oplus (5,10,15) \oplus (5,10,15) \oplus (5,10,15)$$

$$= (30,60,90)$$

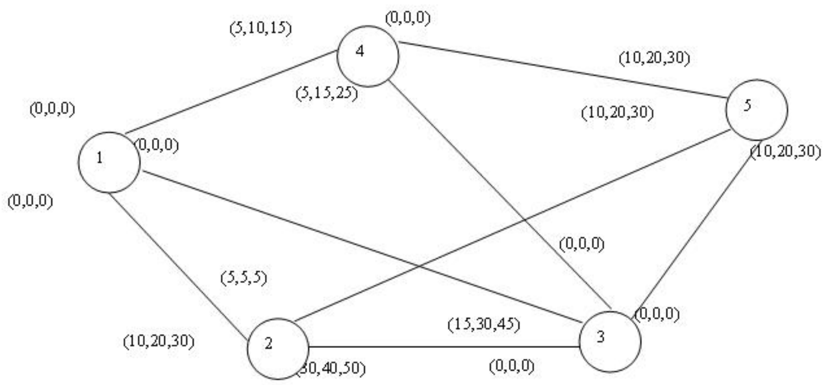

Figure 7. No breakthrough path

7. Result and discussion

The obtained result can be explained as follow:

1) The amount of flow between source and sink is greater than 30 and less than 90 units.

2) Maximum number of persons are in favour that amount of flow will be 60 units.
3) The percentage of favourness for remaining flow can be obtained as follow:

Let x represents the amount of flow, then the percentage of the favourness for $x = \mu_{\tilde{F}}(x) \times 100$ where,

$$\mu_{\tilde{F}}(x) = \begin{cases} \dfrac{(x-30)}{30}, & 30 \le x \le 60 \\ 1 \quad , & x = 60 \\ \dfrac{(90-x)}{30}, & 60 \le x \le 90 \end{cases}$$

As discussed in Section 3, in the optimal network optimal network obtained by using the existing algorithm (Kumar et al. (2009)), shown in Figure 8, the flow from node 1 to node 3, from node 2 to node 5 and from node 5 to node 2 are triangular fuzzy numbers $(-50,0,50)$, $(-35,10,55)$ and $(-20,20,60)$ respectively. Since $\Re(-50,0,50)$ is zero and $\Re(-35,10,55), \Re(-20,20,60)$ are positive so according to properties of ranking functions these flows are non-negative fuzzy numbers. However, there exists negative part in all the obtained triangular fuzzy numbers, which represents that the flow between these nodes may be negative. But the negative quantity of flow has no physical meaning. While it is obvious from the optimal network, shown in Figure 7, obtained by using the improved algorithm there doesn't exist any negative part in any flow i.e., by using the improved algorithm all the shortcomings of the existing algorithm, pointed out in Section 3, are removed.

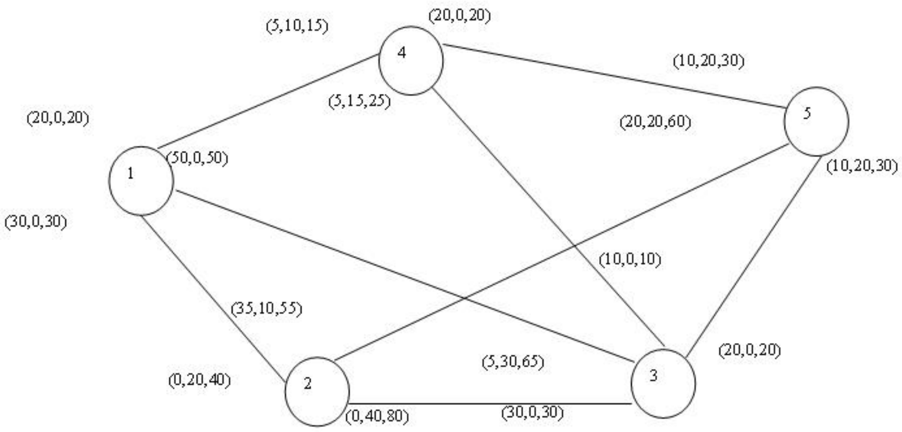

Figure 8. No breakthrough path (Kumar et al 2009)

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