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Ranking Generalized Fuzzy Numbers using Area, Mode, Spreads and Weights

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Abstract: This paper describes a ranking method for ordering fuzzy numbers based on area, mode, spreads and weights of generalized fuzzy numbers. The area used in this method is obtained from the generalized trapezoidal fuzzy number, first by splitting the generalized trapezoidal fuzzy numbers into three plane figures and then calculating the centroids of each plane figure followed by the centroid of these centroids and then finding the area of this centroid from origin which is a process of defuzzification proposed in this paper. This method is simple in evaluation and can rank various types of fuzzy numbers and also crisp numbers which are considered to be a special case of fuzzy numbers.

Keywords: Ranking function; centroid points; generalized trapezoidal fuzzy numbers.

1. Introduction

Ranking fuzzy numbers plays an important role in decision making. Most of the real world problems that exist in nature are fuzzy, than probabilistic or deterministic. Problems in which fuzzy theory is used, like fuzzy risk analysis, fuzzy optimization, etc., at one or the other stage fuzzy numbers must be ranked before an action is taken by a decision maker. As fuzzy numbers are represented by possibility distributions they often overlap with each other and discriminating them is a complex task than discriminating real numbers where a natural order exist between them. An efficient approach for ordering the fuzzy numbers is defuzzification. For this we define a ranking function from the set of all fuzzy numbers *F(R)* to the set of all real numbers '*R*', which maps each fuzzy number into the real line, where a natural order exists. Usually by reducing the whole of any analysis to a single number, much of the information is lost and most of the ranking methods consider only one point of view on comparing fuzzy quantities. Hence an attempt is to be made to minimize this loss.

Since the inception of fuzzy sets by Zadeh [26] in 1965, many authors have proposed different methods for ranking fuzzy numbers. However, due to the complexity of the problem, a method which gives a satisfactory result to all situations is a challenging task. Most of the methods proposed so far are non-discriminating, counter-intuitive and some produce different rankings for the same situation and some methods cannot rank crisp numbers. Ranking fuzzy numbers was first

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proposed by Jain in the year 1976 for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Jain [15, 16] proposed a method using the concept of maximizing set to order the fuzzy numbers and the decision maker considers only the right side membership function. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Yager [23, 24] proposed four indices to order fuzzy quantities in [0, 1]. An Adamo [3] fuzzy decision tree was an important breakthrough in ranking fuzzy numbers. Dubois and Prade [12] proposed a complete set of comparison indices in the frame work of Zadeh's possibility theory. Bortolan and Degani [5] reviewed some of these ranking methods for ranking fuzzy subsets. Chen [7] presented ranking fuzzy numbers with maximizing set and minimizing set. Kim and Park [17] presented a method of ranking fuzzy numbers with index of optimism. Liou and Wang [19] presented ranking fuzzy numbers with integral value. Choobineh and Li [10] presented an index for ordering fuzzy numbers. Since then several methods have been proposed by various researchers which include ranking fuzzy numbers using area compensation by Fortemps and Roubens [13], distance method by Cheng [9]. Wang and Kerre [21, 22] classified the existing ranking procedures into three classes. The first class consists of ranking procedures based on fuzzy mean and spread and second class consists ranking procedures based on fuzzy scoring whereas, the third class consists of methods based on preference relations and concluded that the ordering procedures associated with first class are relatively reasonable for the ordering of fuzzy numbers specially, the ranking procedure presented by Adamo [3] which satisfies all the reasonable properties for the ordering of fuzzy quantities. The methods presented in the second class are not doing well and the methods which belong to class three are reasonable. Later on, ranking fuzzy numbers by area between the centroid point and original point by Chu and Tsao [11], modification of the index of Liou and Wang by Garcia and Lamata [14], fuzzy risk analysis based on ranking of generalized trapezoidal fuzzy numbers by Chen and Chen [6], a new approach for ranking trapezoidal fuzzy numbers by Abbasbandy and Hajjari [1], fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads by Chen and Chen [8] came into existence. Amit Kumar et al. [18] presented a procedure on ranking generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread. Rao and Shankar [20] presented a method on ranking fuzzy numbers using circumcenter of centroids and index of modality.

In this paper a new method is proposed which is based on centroid of centroids to rank fuzzy quantities. In a trapezoidal fuzzy number, first the trapezoid is split into three parts where the first, second and third parts are a triangle, a rectangle and a triangle respectively. Then the centroids of these three parts are calculated followed by the calculation of the centroid of these centroids. Finally, a ranking procedure is defined which is the area between the centroid of centroids and the original point and also uses mode and spreads in those cases where the discrimination is not possible. Most of the ranking procedures proposed in literature, use centroid of trapezoid as reference point, as the centroid is a balancing point of the trapezoid. But the centroid of centroids can be considered to be a more balancing point than the centroid.

The work is organized as follows. Section 2 briefly introduces the basic concepts and definitions of fuzzy numbers. Section 3 presents the proposed new method. In Section 4, some important results like linearity of ranking function and other properties are proved which are useful for proposed approach. In Section 5 the proposed method has been explained with examples which describe the advantages and the efficiency of the

method.

In Section 6 the method demonstrates its robustness by comparing with other methods that exist in literature. Finally, the conclusions of the work are presented in Section 7.

2. Basic Definitions

In this section, some basic definitions are reviewed.

Definition 2.1 Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $f_{\tilde{A}}(x): U \to [0,1]$ where $f_{\tilde{A}}(x)$ is the degree of x in $\tilde{A}$, $\forall x \in U$.

Definition 2.2 A fuzzy set $\tilde{A}$ of universe set U is normal if and only if $Sup_{x \in U} f_{\tilde{A}}(x) = 1$

Definition 2.3 A fuzzy set $\tilde{A}$ of universe set U is convex if and only if
$f_{\tilde{A}}(\lambda x + (1-\lambda) y) \geq \min\left(f_{\tilde{A}}(x), f_{\tilde{A}}(y)\right), \forall x, y \in U$ and $\lambda \in [0,1]$

Definition 2.4 A fuzzy set $\tilde{A}$ of universe set U is a fuzzy number iff $\tilde{A}$ is normal and convex on U.

Definition 2.5 A real fuzzy number $\tilde{A}$ is described as any fuzzy subset of the real line R with membership function $f_{\tilde{A}}(x)$ possessing the following properties:

(i) $f_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval $[0, w]; 0 < w \leq 1$

(ii) $f_{\tilde{A}}(x)$=0, for all $x \in (-\infty, a] \cup [d, \infty)$

(iii) $f_{\tilde{A}}(x)$ is strictly increasing on $[a,b]$ and strictly decreasing on $[c,d]$

(iv) $f_{\tilde{A}}(x)$=1, for all $x \in [b,c]$ where a, b, c, d are real numbers.

Definition 2.6 The membership function of the real fuzzy number $\tilde{A}$ is given by

$$f_{\tilde{A}}(x) = \begin{cases} f_{\tilde{A}}^{L}(x), & a \leq x \leq b, \\ w, & b \leq x \leq c, \\ f_{\tilde{A}}^{R}(x), & c \leq x \leq d, \\ 0, & otherwise, \end{cases}$$

where $0 < w \leq 1$ is a constant, a, b, c, d are real numbers and $f_{\tilde{A}}^{L}: [a,b] \to [0,w]$ $f_{\tilde{A}}^{R}: [c,d] \to [0,w]$ are two strictly monotonic and continuous functions from R to the closed interval $[0,w]$. It is customary to write a fuzzy number as $\tilde{A} = (a,b,c,d;w)$. If $w = 1$, then

$\tilde{A} = (a,b,c,d;1)$ is a normalized fuzzy number, otherwise $\tilde{A}$ is said to be a generalized or non-normal fuzzy number if $0 < w < 1$.

If the membership function is $f_{\tilde{A}}(x)$ piecewise linear, then $\tilde{A}$ is said to be a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number is given by:

$$f_{\tilde{A}}(x) = \begin{cases} \dfrac{w(x-a)}{b-a}, & a \le x \le b, \\ w, & b \le x \le c, \\ \dfrac{w(x-d)}{c-d}, & c \le x \le d, \\ 0, & otherwise. \end{cases}$$

If w=1, then $\tilde{A} = (a,b,c,d;1)$ is a normalized trapezoidal fuzzy number, otherwise $\tilde{A}$ is a generalized or non normal trapezoidal fuzzy number if $0 < w < 1$.

The image of $\tilde{A} = (a,b,c,d;w)$ is given by $-\tilde{A} = (-d,-c,-b,-a;w)$.
As a particular case if $b = c$, the trapezoidal fuzzy number reduces to a triangular fuzzy number given by $\tilde{A} = (a,b,d;w)$. The value of 'b' corresponds with the mode or core and [a, d] with the support. If w=1, then $\tilde{A} = (a,b,d)$ is a normalized triangular fuzzy number, otherwise $\tilde{A}$ is a generalized or non normal triangular fuzzy number if $0 < w < 1$.

Definition 2.7 If $\tilde{A} = (a_1,b_1,c_1,d_1;w_1)$ and $\tilde{B} = (a_2,b_2,c_2,d_2;w_2)$ are two generalized trapezoidal fuzzy numbers, then

(i) $\tilde{A} \oplus \tilde{B} = \left(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2)\right)$

(ii) $\tilde{A} \Theta \tilde{B} = \left(a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2)\right)$

(iii) $k\tilde{A} = (ka_1, kb_1, kc_1, kd_1; w_1); k > 0$

(iv) $k\tilde{A} = (kd_1, kc_1, kb_1, ka_1; w_1); k < 0$

3. Proposed Method

The centroid of a trapezoid is considered to be the balancing point of the trapezoid (Fig.1). Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC) and again a triangle (CQD) respectively. Let the centroids of the three plane figures be $G_1, G_2 \& G_3$ respectively.

The centroid of these centroids $G_1, G_2 \& G_3$ is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each centroid point G_1 of triangle APB, G_2 of rectangle BPQC and G_3 of triangle CQD are balancing points of each individual plane figure and the centroid of these centroid points is a much more balancing point for a generalized trapezoidal fuzzy number.

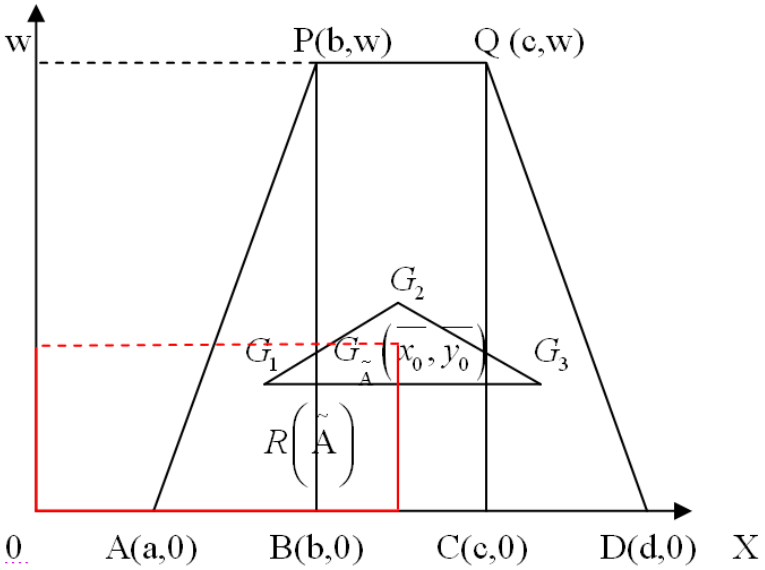

Figure 1. Centroid of centroids

Consider a generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ (Figure 1)

The centroids of the three plane figures are:

$$G_1 = \left(\frac{a+2b}{3}, \frac{w}{3}\right); \quad G_2 = \left(\frac{b+c}{2}, \frac{w}{2}\right)$$

and

$$G_3 = \left(\frac{2c+d}{3}, \frac{w}{3}\right) \text{ respectively.}$$

Equation of the line $\overleftrightarrow{G_1G_3}$ is $y = \frac{w}{3}$ and G_2 does not lie on the line $\overleftrightarrow{G_1G_3}$.

Therefore, G_1, G_2 and G_3 are non-collinear and they form a triangle.

We define the centroid $G_{\tilde{A}}(\overline{x_0}, \overline{y_0})$ of the triangle with vertices G_1, G_2 and G_3 of the generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ as

$$G_{\tilde{A}}(\overline{x_0}, \overline{y_0}) = \left(\frac{2a+7b+7c+2d}{18}, \frac{7w}{18}\right) \quad (1)$$

As a special case, for triangular fuzzy number $\tilde{A} = (a,b,d;w)$ i.e., $c = b$ the centroid of centroids is given by

$$G_{\tilde{A}}(\overline{x_0}, \overline{y_0}) = \left(\frac{a+7b+d}{9}, \frac{7w}{18}\right) \quad (2)$$

The ranking function of the generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as:

$$R(\tilde{A}) = \overline{x_0} \cdot \overline{y_0} = \left(\frac{2a+7b+7c+2d}{18}\right)\left(\frac{7w}{18}\right) \quad (3)$$

This is the area between the centroid of the centroids $G_{\tilde{A}}(\overline{x_0}, \overline{y_0})$ as defined in (1) and the original point.

The mode of the generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ is defined as:

$$mode = \frac{1}{2}\int_0^w (b+c)\,dx = \frac{w}{2}(b+c) \quad (4)$$

The spread of the generalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;w)$ is defined as:

$$spread = \int_0^w (d-a)\,dx = w(d-a) \quad (5)$$

The left spread of the generalized trapezoi-

dal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is defined as:

$$\textit{left spread} = \int_0^w (b-a)\,dx = w(b-a) \qquad (6)$$

The right spread of the generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is defined as:

$$\textit{right spread} = \int_0^w (d-c)\,dx = w(d-c) \qquad (7)$$

Using the above definitions we now define the ranking procedure of two generalized trapezoidal fuzzy numbers.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers. The working procedure to compare $\tilde{A}$ *and* $\tilde{B}$ is as follows:

Step 1: Find $R\left(\tilde{A}\right)$ and $R\left(\tilde{B}\right)$

Case (i) If $R\left(\tilde{A}\right) > R\left(\tilde{B}\right)$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $R\left(\tilde{A}\right) < R\left(\tilde{B}\right)$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$ comparison is not possible, then go to step 2.

Step 2: Find mode$\left(\tilde{A}\right)$ and mode$\left(\tilde{B}\right)$

Case (i) If mode$\left(\tilde{A}\right)$ > mode$\left(\tilde{B}\right)$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If mode$\left(\tilde{A}\right)$ <mode$\left(\tilde{B}\right)$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If mode$\left(\tilde{A}\right)$ = mode$\left(\tilde{B}\right)$ comparison is not possible, then go to step 3.

Step 3: Find spread$\left(\tilde{A}\right)$ and spread$\left(\tilde{B}\right)$

Case (i) If spread$\left(\tilde{A}\right)$ > spread$\left(\tilde{B}\right)$ then $\tilde{A} \prec \tilde{B}$

Case (ii) If spread$\left(\tilde{A}\right)$ < spread$\left(\tilde{B}\right)$ then $\tilde{A} \succ \tilde{B}$

Case (iii) If spread$\left(\tilde{A}\right)$ = spread$\left(\tilde{B}\right)$ comparison is not possible, then go to step 4.

Step 4: Find left spread$\left(\tilde{A}\right)$ and left spread$\left(\tilde{B}\right)$

Case (i) If left spread$\left(\tilde{A}\right)$ > left spread$\left(\tilde{B}\right)$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If left spread$\left(\tilde{A}\right)$ < left spread$\left(\tilde{B}\right)$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If left spread$\left(\tilde{A}\right)$ = left spread$\left(\tilde{B}\right)$ comparison is not possible, then go to step 5.

Step 5: Examine w_1 and w_2

Case (i) If $w_1 > w_2$ then $\tilde{A} \succ \tilde{B}$

Case (ii) If $w_1 < w_2$ then $\tilde{A} \prec \tilde{B}$

Case (iii) If $w_1 = w_2$ then $\tilde{A} \approx \tilde{B}$

4. Some important results

In this section some important results which are the basis for defining the ranking procedure in section 3 are discussed and proved.

Proposition 4.1 The ranking function defined in section 3 by means of equation (3) is a linear function for normalized trapezoidal fuzzy number $\tilde{A} = (a,b,c,d;1)$ i.e. $R\left(\tilde{A}\right) = \left(\frac{2a+7b+7c+2d}{18}\right)\left(\frac{7}{18}\right)$

If $\tilde{A} = (a_1,b_1,c_1,d_1)$ and $\tilde{B} = (a_2,b_2,c_2,d_2)$ are two normalized trapezoidal fuzzy numbers, then

(i) $R\left(k_1\tilde{A} \oplus k_2\tilde{B}\right) = k_1R\left(\tilde{A}\right) \oplus k_2R\left(\tilde{B}\right); k_1,k_2 \in R$

(ii) $R\left(-\tilde{A}\right) = -R\left(\tilde{A}\right)$

(iii) $R\left((\tilde{A}) \oplus (-\tilde{A})\right) = 0$

Proof (i):

case (i) Let $k_1,k_2 > 0$

$$k_1\tilde{A} \oplus k_2\tilde{B} = \left(k_1a_1 + k_2a_2, k_1b_1 + k_2b_2, k_1c_1 + k_2c_2, k_1d_1 + k_2d_2\right)$$

$$R\left(k_1\tilde{A} \oplus k_2\tilde{B}\right) = \left(\frac{2(k_1a_1 + k_2a_2) + 7(k_1b_1 + k_2b_2) + 7(k_1c_1 + k_2c_2) + 2(k_1d_1 + k_2d_2)}{18}\right)\left(\frac{7}{18}\right)$$

$$= \left(\frac{k_1(2a_1 + 7b_1 + 7c_1 + 2d_1) + k_2(2a_2 + 7b_2 + 7c_2 + 2d_2)}{18}\right)\left(\frac{7}{18}\right)$$

$$= k_1R\left(\tilde{A}\right) \oplus k_2R\left(\tilde{B}\right)$$

Similarly the result can be proved for case (ii) $k_1 > 0, k_2 < 0$ and case (iii) $k_1 < 0, k_2 > 0$.

Proof (ii): Let $\tilde{A} = (a,b,c,d) \Rightarrow -\tilde{A} = (-d,-c,-b,-a)$

$$R\left(-\tilde{A}\right) = \left(\frac{-2d - 7c - 7b - 2a}{18}\right)\left(\frac{7}{18}\right) = -\left(\frac{2a + 7b + 7c + 2d}{18}\right)\left(\frac{7}{18}\right) = -R\left(\tilde{A}\right)$$

Proof (iii): $R\left((\tilde{A}) \oplus (-\tilde{A})\right) = R\left(\tilde{A}\right) \oplus R\left(-\tilde{A}\right)$ **(by (i))**

$= R\left(\tilde{A}\right)\Theta R\left(\tilde{A}\right)$ **(by (ii))**

=0.

Proposition 4.2 Let $\tilde{A} = (a_1,b_1,c_1,d_1;w_1)$ and $\tilde{B} = (a_2,b_2,c_2,d_2;w_2)$ be two generalized trapezoidal fuzzy numbers such that $R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$; mode $\left(\tilde{A}\right)$ = mode $\left(\tilde{B}\right)$;

$\text{spread}\left(\tilde{\text{A}}\right) = \text{spread}\left(\tilde{\text{B}}\right)$ then

(i) $\text{left spread}\left(\tilde{\text{A}}\right) > \text{left spread}\left(\tilde{\text{B}}\right) \Leftrightarrow w_1 b_1 > w_2 b_2$

(ii) $\text{left spread}\left(\tilde{\text{A}}\right) < \text{left spread}\left(\tilde{\text{B}}\right) \Leftrightarrow w_1 b_1 < w_2 b_2$

(iii) $\text{left spread}\left(\tilde{\text{A}}\right) = \text{left spread}\left(\tilde{\text{B}}\right) \Leftrightarrow w_1 b_1 = w_2 b_2$

Proof: From the assumptions

$$R\left(\tilde{\text{A}}\right) = R\left(\tilde{\text{B}}\right) \Rightarrow w_1(2a_1 + 7b_1 + 7c_1 + 2d_1) = w_2(2a_2 + 7b_2 + 7c_2 + 2d_2) \quad (8)$$

$$\text{mode}\left(\tilde{\text{A}}\right) = \text{mode}\left(\tilde{\text{B}}\right) \Rightarrow w_1(b_1 + c_1) = w_2(b_2 + c_2) \quad (9)$$

$$\text{spread}\left(\tilde{\text{A}}\right) = \text{spread}\left(\tilde{\text{B}}\right) \Rightarrow w_1(d_1 - a_1) = w_2(d_2 - a_2) \quad (10)$$

Solving Equations (8), (9) and (10) we get

$w_1 a_1 = w_2 a_2$

$w_1 d_1 = w_2 d_2$

Now to prove (i):

$\text{left spread}\left(\tilde{\text{A}}\right) > \text{left spread}\left(\tilde{\text{B}}\right)$

$\Leftrightarrow w_1(b_1 - a_1) > w_2(b_2 - a_2)$

$\Leftrightarrow w_1 b_1 > w_2 b_2 (\because w_1 a_1 = w_2 a_2)$

Now to prove (ii):

$\text{left spread}\left(\tilde{\text{A}}\right) < \text{left spread}\left(\tilde{\text{B}}\right)$

$\Leftrightarrow w_1(b_1 - a_1) < w_2(b_2 - a_2)$

$\Leftrightarrow w_1 b_1 < w_2 b_2 (\because w_1 a_1 = w_2 a_2)$

Now to prove (iii):

$\text{left spread}\left(\tilde{\text{A}}\right) = \text{left spread}\left(\tilde{\text{B}}\right)$

$\Leftrightarrow w_1(b_1 - a_1) = w_2(b_2 - a_2)$

$\Leftrightarrow w_1 b_1 = w_2 b_2 (\because w_1 a_1 = w_2 a_2)$

Corollary 4.2 All the results of proposition 4.2 also hold for right spread.

Proposition 4.3 Let $\tilde{\text{A}} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{\text{B}} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers such that

$R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$; mode$\left(\tilde{A}\right)$ = mode$\left(\tilde{B}\right)$; spread$\left(\tilde{A}\right)$ = spread$\left(\tilde{B}\right)$ then

(i) left spread$\left(\tilde{A}\right)$ > left spread$\left(\tilde{B}\right)$ $\Leftrightarrow$ right spread$\left(\tilde{A}\right)$ > right spread$\left(\tilde{B}\right)$

(ii) left spread$\left(\tilde{A}\right)$ < left spread$\left(\tilde{B}\right)$ $\Leftrightarrow$ right spread$\left(\tilde{A}\right)$ < right spread$\left(\tilde{B}\right)$

(iii) left spread$\left(\tilde{A}\right)$ = left spread$\left(\tilde{B}\right)$ $\Leftrightarrow$ right spread$\left(\tilde{A}\right)$ = right spread$\left(\tilde{B}\right)$

Proof: From proposition 4.2, for the above assumptions we have

$w_1(b_1 + c_1) = w_2(b_2 + c_2)$

$w_1 a_1 = w_2 a_2$

$w_1 d_1 = w_2 d_2$

Now to prove (i):

left spread$\left(\tilde{A}\right)$ > left spread$\left(\tilde{B}\right)$

$\Leftrightarrow w_1 b_1 - w_2 b_2 > 0 (\because w_1 a_1 = w_2 a_2)$ (from proposition 4.2)

$\Leftrightarrow w_2 c_2 - w_1 c_1 > 0 (\because w_1(b_1 + c_1) = w_2(b_2 + c_2) \Rightarrow w_1 b_1 - w_2 b_2 = w_2 c_2 - w_1 c_1)$

$\Leftrightarrow -w_1 c_1 > -w_2 c_2$

$\Leftrightarrow w_1(d_1 - c_1) > w_2(d_2 - c_2) (\because w_1 d_1 = w_2 d_2)$

$\Leftrightarrow$ right spread$\left(\tilde{A}\right)$ > right spread$\left(\tilde{B}\right)$

Similarly (ii) and (iii) can be proved.

5. Numerical Examples

In this section, the proposed method is first explained by ranking some fuzzy numbers.

Example 5.1

Let $\tilde{A} = (3,5,7;1)$ and $\tilde{B} = \left(4,5,\frac{51}{8};1\right)$

Step 1:

$G_{\tilde{A}}\left(\overline{x_0}, \overline{y_0}\right) = (5, 0.3888)$ and $G_{\tilde{B}}\left(\overline{x_0}, \overline{y_0}\right) = (5.0416, 0.3888)$

$R\left(\tilde{A}\right) = 1.944$ *and* $R\left(\tilde{B}\right) = 1.960$

Since $R\left(\tilde{A}\right) < R\left(\tilde{B}\right) \Rightarrow \tilde{A} \prec \tilde{B}$

Example 5.2

Let $\tilde{A} = (0,1,2;1), \tilde{B} = \left(\frac{1}{5},1,\frac{7}{4};1\right)$

Step 1:

$G_{\underset{A}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (1, 0.3888)$ and $G_{\underset{B}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (0.9944, 0.3888)$ $\quad R\left(\tilde{A}\right) = 0.3888$ *and* $R\left(\tilde{B}\right) = 0.3866$

Since $R\left(\tilde{A}\right) > R\left(\tilde{B}\right) \Rightarrow \tilde{A} \succ \tilde{B}$

Example 5.3

Let $\tilde{A} = (0,1,2;1) \Rightarrow -\tilde{A} = (-2,-1,0;1)$ and $\tilde{B} = \left(\frac{1}{5},1,\frac{7}{4};1\right) \Rightarrow -\tilde{B} = \left(\frac{-7}{4},-1,\frac{-1}{5};1\right)$

Step 1:

$G_{\underset{-A}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (-1, 0.3888)$ and $G_{\underset{-B}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (-0.9944, 0.3888)$

$R\left(-\tilde{A}\right) = -0.3888$ *and* $R\left(-\tilde{B}\right) = -0.3866$

Since $R\left(-\tilde{A}\right) < R\left(-\tilde{B}\right) \Rightarrow -\tilde{A} \prec -\tilde{B}$

From examples 5.2 and 5.3 we see that the proposed method can rank fuzzy numbers and their images as it is proved that $\tilde{A} \succ \tilde{B} \Rightarrow -\tilde{A} \prec -\tilde{B}$.

Example 5.4

Let $\tilde{A} = (0.1, 0.3, 0.5;1)$ and $\tilde{B} = (0.2, 0.3, 0.4;1)$

Step 1:

$G_{\underset{A}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (0.3, 0.3888)$ and $G_{\underset{B}{\sim}}\left(\overline{x_0},\overline{y_0}\right) = (0.3, 0.3888)$

Since $R\left(\tilde{A}\right) = 0.1166$ *and* $R\left(\tilde{B}\right) = 0.1166 \Rightarrow R\left(\tilde{A}\right) = R\left(\tilde{B}\right)$, so go to step 2.

Step 2: mode$\left(\tilde{A}\right)$ = 0.3 and mode$\left(\tilde{B}\right)$ = 0.3

Since mode$\left(\tilde{A}\right)$ = mode$\left(\tilde{B}\right)$, so go to step 3.

Step 3: spread$\left(\tilde{A}\right)$ = 0.4 and spread$\left(\tilde{B}\right)$ = 0.2

Since spread$\left(\tilde{A}\right)$ > spread$\left(\tilde{B}\right) \Rightarrow \tilde{A} \prec \tilde{B}$

Example 5.5

Let $\tilde{A} = (0.1, 0.3, 0.5; 0.8), \tilde{B} = (0.1, 0.3, 0.5; 1)$

Step 1:

$G_{\tilde{A}}(\overline{x_0}, \overline{y_0}) = (0.3, 0.30004)$ and $G_{\tilde{B}}(\overline{x_0}, \overline{y_0}) = (0.3, 0.3888)$

$R(\tilde{A}) = 0.0900$ *and* $R(\tilde{B}) = 0.1166$

As $R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} \prec \tilde{B}$

From example 5.5 it is clear that the proposed method can rank fuzzy numbers with different height and same spreads.

Example 5.6

Let $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$ and $\tilde{B} = (0.1, 0.3, 0.5; 1)$

Then $G_{\tilde{A}}(\overline{x_0}, \overline{y_0}) = (0.3, 0.3888)$ and $G_{\tilde{B}}(\overline{x_0}, \overline{y_0}) = (0.3, 0.3888)$

Since $R(\tilde{A}) = 0.1166$ *and* $R(\tilde{B}) = 0.1166 \Rightarrow R(\tilde{A}) = R(\tilde{B})$, so go to step 2.

Step 2: mode$(\tilde{A})$ = 0.3 and mode$(\tilde{B})$ = 0.3

Since mode$(\tilde{A})$ = mode$(\tilde{B})$, so go to step 3.

Step 3: spread$(\tilde{A})$ = 0.4 and spread$(\tilde{B})$ = 0.4

Since spread$(\tilde{A})$ = spread$(\tilde{B})$, so go to step 4.

Step 4: left spread$(\tilde{A})$ = 0.1 and left spread$(\tilde{B})$ = 0.2

Since left spread$(\tilde{A})$ < left spread$(\tilde{B}) \Rightarrow \tilde{A} \prec \tilde{B}$.

6. Results and discussion

In this section the advantages of the proposed method are shown by comparing with other existing methods in literature, where the methods failed to discriminate fuzzy numbers. The results are shown in Table 1, Table 2 and Table 3.

Example 6.1

Consider two fuzzy numbers *A*= (1, 4, 5) and *B*= (2, 3, 6)

By Liou and Wang method [17] it is clear that the two fuzzy numbers are equal for all the

decision maker's as $I_T^{\alpha}(A) = 4.5\alpha + (1-\alpha)2.5$ and $I_T^{\alpha}(B) = 4.5\alpha + (1-\alpha)2.5$ which is not even true by intuition. By using our method we have:

$G_A\left(\overline{x_0}, \overline{y_0}\right) = (3.7777, 0.3888)$ and $G_B\left(\overline{x_0}, \overline{y_0}\right) = (3.2222, 0.3888)$

$R\left(\tilde{\text{A}}\right) = 1.4687$ *and* $R\left(\tilde{\text{B}}\right) = 1.2527$

Since $R\left(\tilde{\text{A}}\right) > R\left(\tilde{\text{B}}\right) \Rightarrow \tilde{\text{A}} \succ \tilde{\text{B}}$

Example 6.2

Let $\tilde{\text{A}} = (0.1, 0.3, 0.5; 1)$ and $\tilde{\text{B}} = (1,1,1,1;1)$

Cheng [7] proposed a ranking function which is the distance from centroid point and the original point where as Chu and Tsao [11] proposed a ranking function which is the area between the centroid point and original point. Their centroid formulae are given by:

$$\left(\frac{w\left(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab\right) + 3\left(c^2 - b^2\right)}{3w\left(d - c + b - a\right) + 6\left(c - b\right)}, \frac{w}{3}\left(1 + \frac{(b+c) - (a+d)(1-w)}{(b+c-a-d) + 2(a+d)w}\right)\right)$$

And $$\left(\frac{w\left(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab\right) + 3\left(c^2 - b^2\right)}{3w\left(d - c + b - a\right) + 6\left(c - b\right)}, \frac{w}{3}\left(1 + \frac{b+c}{a+b+c+d}\right)\right)$$

Both these centroid formulae cannot rank crisp numbers which are a special case of fuzzy numbers as, it can be seen from the above formulae that the denominator in the first coordinate of their centroid formulae is zero and hence, centroid of crisp numbers are undefined for their formulae. By using our method we have:

$G_{\tilde{\text{A}}}\left(\overline{x_0}, \overline{y_0}\right) = (0.3, 0.3888)$ and $G_{\tilde{\text{B}}}\left(\overline{x_0}, \overline{y_0}\right) = (1, 0.3888)$

$R\left(\tilde{\text{A}}\right) = 0.1166$ *and* $R\left(\tilde{\text{B}}\right) = 0.3888$

Since $R\left(\tilde{\text{A}}\right) < R\left(\tilde{\text{B}}\right) \Rightarrow \tilde{\text{A}} \prec \tilde{\text{B}}$

From this example it is proved that the proposed method can rank crisp numbers whereas, Cheng's method [9] and Chu and Tsao's method [11] failed to do so.

Example 6.3

Consider four fuzzy numbers

$A_1 = (0.1, 0.2, 0.3; 1)$; $A_2 = (0.2, 0.5, 0.8; 1)$; $A_3 = (0.3, 0.4, 0.9; 1)$; $A_4 = (0.6, 0.7, 0.8; 1)$ as shown in Fig. 2 which were ranked earlier by Yager [24], Fortemps and Roubens [13], Liou and Wang [19] and Chen [7]. The results are shown in Table 1.

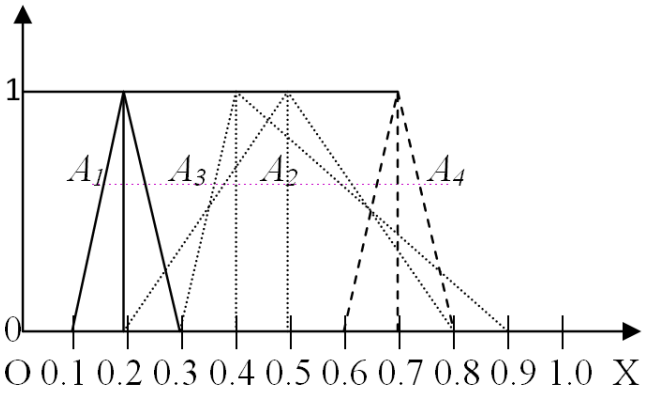

Figure 2. A_1= (0.1, 0.2, 0.3;1), A_2 = (0.2, 0.5, 0.8;1)
A_3= (0.3, 0.4, 0.9;1), A_4 = (0.6, 0.7, 0.8;1)

Table 1. Comparison of various ranking methods

Method\Fuzzy number →		A_1	A_2	A_3	A_4	Ranking order
Yager [24]		0.20	0.50	0.50	0.70	$A_4 \succ A_2 \approx A_3 \succ A_1$
Fortemps & Roubens[13]		0.20	0.50	0.50	0.70	$A_4 \succ A_2 \approx A_3 \succ A_1$
Liou & Wang[19]	$\alpha = 1$	0.25	0.65	0.65	0.75	$A_4 \succ A_2 \approx A_3 \succ A_1$
	$\alpha = 0.5$	0.20	0.50	0.50	0.70	$A_4 \succ A_2 \approx A_3 \succ A_1$
	$\alpha = 0$	0.15	0.35	0.35	0.65	$A_4 \succ A_2 \approx A_3 \succ A_1$
Chen [7]	$\beta = 1$	-0.20	0.00	0.00	-0.20	$A_2 \approx A_3 \succ A_1 \approx A_4$
	$\beta = 0.5$	-0.20	0.00	0.00	-0.20	$A_2 \approx A_3 \succ A_1 \approx A_4$
	$\beta = 0$	-0.20	0.00	0.00	-0.20	$A_2 \approx A_3 \succ A_1 \approx A_4$
Proposed method		0.0777	0.1914	0.1727	0.2721	$A_4 \succ A_2 \succ A_3 \succ A_1$

It can be seen from Table 1 that none of the above methods able to discriminate the given fuzzy numbers, except the proposed method. Yager [24] and Fortemps [13] methods failed to discriminate the fuzzy numbers A_2 and A_3 whereas, the methods of Liou [19] and Chen [7] cannot discriminate the fuzzy numbers A_2, A_3 and A_1, A_4.
By using our method we have:

$$G_{A_1}\left(\overline{x_0}, \overline{y_0}\right) = (0.2, 0.3888) \quad , \quad G_{A_2}\left(\overline{x_0}, \overline{y_0}\right) = (0.5, 0.3888) \quad , \quad G_{A_3}\left(\overline{x_0}, \overline{y_0}\right) = (0.4444, 0.3888)$$

$$G_{A_4}\left(\overline{x_0}, \overline{y_0}\right) = (0.7, 0.3888)$$

Since $R(A_1) = 0.0777, R(A_2) = 0.1944, R(A_3) = 0.1727, R(A_4) = 0.2721 \Rightarrow A_4 \succ A_2 \succ A_3 \succ A_1$

Example 6.4

In this we consider six sets of fuzzy numbers available in literature from [6] which is shown in Fig. 3 and the comparative study with various approaches is presented in Table 2.

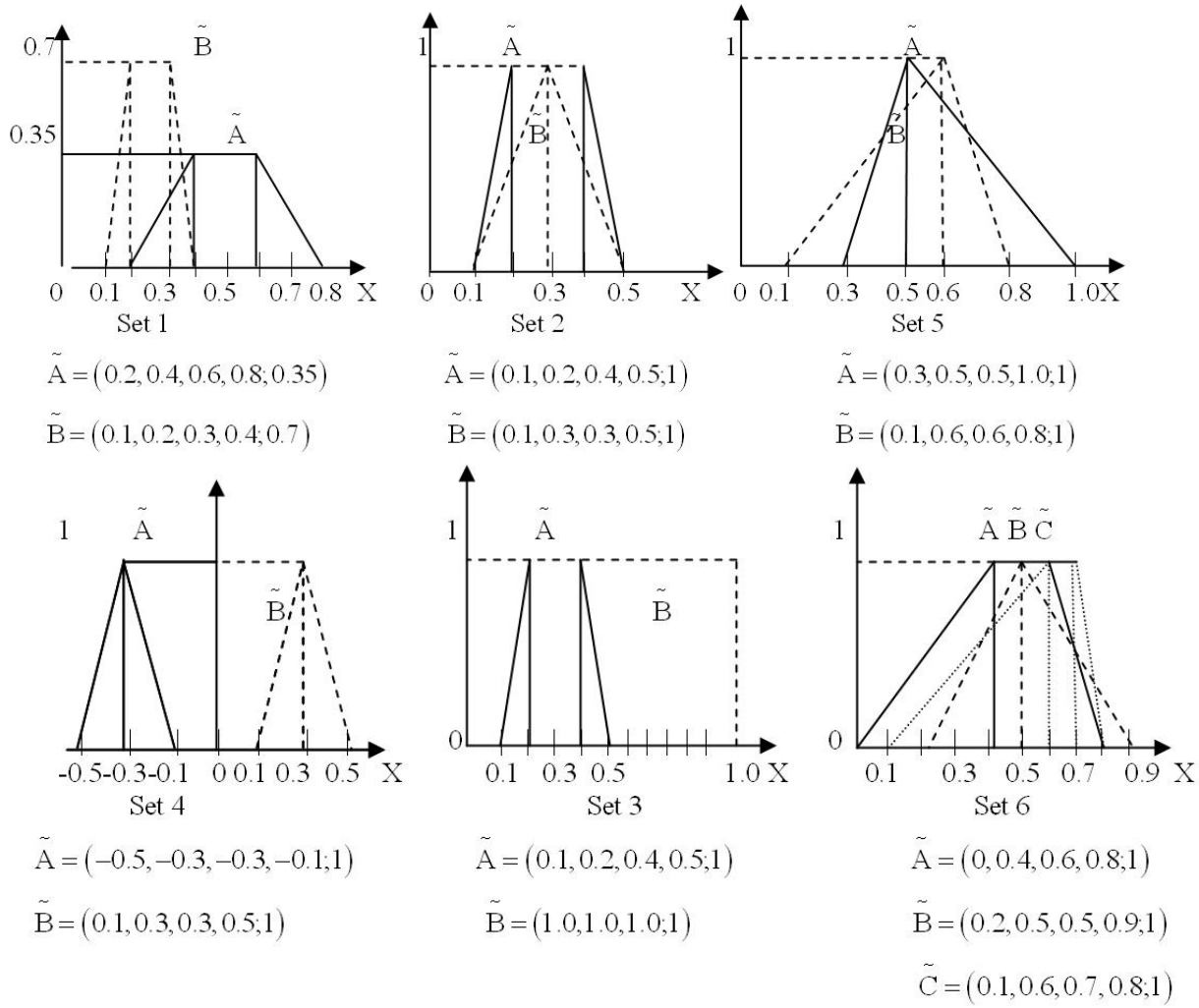

Figure 3. Six sets of fuzzy numbers (Chen & Chen [6])

Table 2. A Comparison of the ranking results for different approaches

Methods	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
Cheng [9]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	Not Comparable	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chu and Tsao [11]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	Not Comparable	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen and Chen [6]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Abbasbandy and Hajjari [1]	Not Comparable	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \approx \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen and Chen [8]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Kumar et al.[18]	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Proposed method	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$

Example 6.5

In this we consider four sets of fuzzy numbers available in a literature from [1] which is shown in Figure 4 and the comparative study is presented in Table 3.

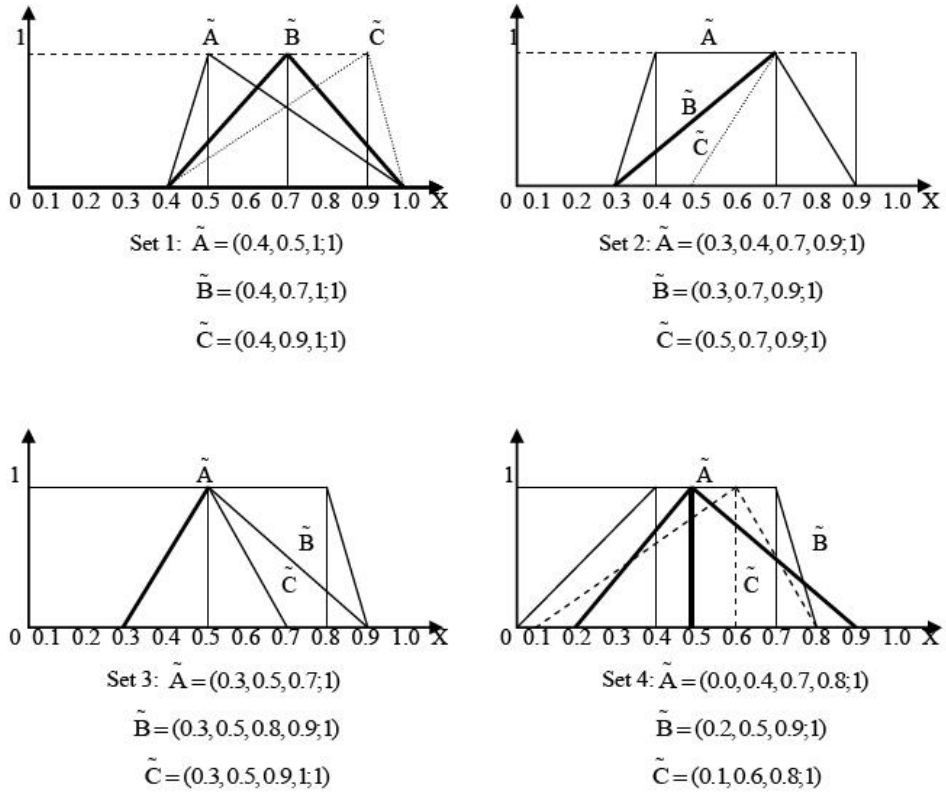

Figure 4. Four sets of fuzzy numbers (Abbasbandy and Hajjari [1])

Table 3. A Comparison of the ranking results for different approaches

Authors/Methods	Set 1	Set 2	Set 3	Set 4
Abbasbandy and Hajjari [1]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$B \prec A \prec C$
Sign distance method p=1[2]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Sign distance method p=2 [2]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Choobineh and Li [10]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Yager [24]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Chen [6]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Baldwin and Guild [4]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Chu and Tsao [11]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Yao and Wu [25]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Cheng distance [9]	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Cheng CV uniform distribution [9]	$B \prec C \prec A$	$C \prec B \prec A$	$A \prec C \prec B$	$B \prec C \prec A$
Cheng CV proportional distribution [9]	$B \prec C \prec A$	$C \prec B \prec A$	$A \prec C \prec B$	$B \prec C \prec A$
Proposed Method	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$B \prec A \prec C$

7. Conclusions and future work

This paper proposes a method that ranks fuzzy numbers which is simple and concrete. This method ranks trapezoidal as well as triangular fuzzy numbers and their images. This method also ranks crisp numbers which are special case of fuzzy numbers whereas some methods proposed in literature cannot rank crisp numbers. This method which is simple and easier in calculation not only gives satisfactory results to well defined problems, but also gives a correct ranking order to problems which are not well defined. Comparative examples are used to illustrate the advantages of the proposed method. Application of this ranking procedure in various decision making problems such as, fuzzy risk analysis and in fuzzy optimization like network analysis is left as future work.

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