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Asymptotic - Numerical Method for Third-Order Singular Perturbation Problems

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Abstract: In this paper, a class of third order singularly perturbed boundary value problems with suitable boundary conditions is considered. The third order boundary value problem is transformed to asymptotically equivalent second order boundary value problem. This problem is solved efficiently by using fitted Numerov method. Linear and non-linear examples are solved to illustrate the method and relative errors with L_2-norms are presented to support the method.

Keywords: Fitted numerov method; boundary value problem; tridiagonal system; fitting factor; relative error.

1. Introduction

The numerical methods for second-order singularly perturbed differential equations have been extensively analysed in the last twenty years, whereas only few results on higher-order problems are found in the literature. Analytical treatment for higher-order non-linear ordinary differential equations, which have important applications in Fluid Dynamics, are available in [1-5]. The classification of singularly perturbed higher-order problems depends on how the order of the original differential operator is affected if one sets $\varepsilon = 0$. When the order is reduced by one, we say the problem is of convection-diffusion type and of a reaction-diffusion type if the order is reduced by two. Here ε is a small positive parameter multiplying the highest derivative of the differential equation.

In the literature only very few works have been reported for higher order problems.

Howes [4] has considered a class of third order singular perturbation problems and discussed existence and asymptotic behaviour of the solution. An iterative method for higher order problems is discussed in [3]. A. Ramesh babu and N. Ramanujam [8] considered singularly perturbed boundary value problems for third and fourth order ordinary differential equations with discontinuous source term and a small positive parameter multiplying the highest derivative and presented a computational method named as an asymptotic finite element method for solving these systems.

In this paper, a class of third order singularly perturbed boundary value problems with suitable boundary conditions is considered. The third order boundary value problem is transformed to asymptotically equivalent second order boundary value problem.

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This problem is solved efficiently by using fitted Numerov method. Linear and non-linear examples are solved to illustrate the method and relative errors with L_2 -norms are presented to support the method.

2. Description of the method

We consider a third order boundary value problem of the type

$$\begin{aligned} &\varepsilon^2 y''' = f(y)y' + g(x,y),\ a < x < b, \\ &y(a,\varepsilon) = A,\ y'(a,\varepsilon) = C,\ y(b,\varepsilon) = B \end{aligned} \qquad (1)$$

where *f(y)* and *g(x, y)* have a sufficient number of continues derivatives and ε is small positive constant. Since ε is small, in order to study the behavior of solutions of (1) as $\varepsilon \to 0^+$, we should first examine the solution of the corresponding reduced problem

$$f(y)y' + g(x,y) = 0,\ a < x < b,\ y(a) = A \qquad (2)$$

Let $y_0(x)$ be the smooth solution of this problem and that $y_0(x)$ is stable in the sense that $f(y_0(x)) \geq m^2 > 0$ in $[a,\ b]$ for a positive constant *m*. This function $y_0(x)$ is our candidate for an approximate solution of the problem (1).

We now proceed to replace (1) with an asymptotically equivalent second order problem.

We write the problem (1) as

$$\varepsilon^2 y''' = [F(y)]' + g(x, y_0(x)) \qquad (3)$$

We integrate this equation to obtain the corresponding second-order problem

$$\begin{aligned} &\varepsilon^2 y'' = F(y) + G(x) \equiv H(x,y),\quad a < x < b, \\ &y'(a,\varepsilon) = C,\ y(b,\varepsilon) = B, \end{aligned} \qquad (4)$$

where *G(x)* is an antiderivative of $g(x, y_0(x))$.

In order to ensure that $y_0(x)$ is a solution of the reduced equation, $H(x, y_0(x)) = F(y_0(x)) + G(x) = 0,$ of the problem (3), we choose F and G so that $F(A) + G(a) = 0,$ that is, $G(x) = -F(y_0(x))$ and so $H(x, y_0(x)) = F(y) - F(y_0(x)).$

In terms of the function H, we have (by assumption) that

$$H(x, y_0(x)) = 0 \text{ and } \frac{\partial H(x, y_0(x))}{\partial y} = f(y_0(x)) \geq m^2 > 0 \text{ in } [a, b].$$

If in addition, we have that $\frac{\partial H(x,y)}{\partial y} = f(y) \geq m^2 > 0$ for all values of y between y(b) and B, then it is known [5, part II]) that the problem (3) has a solution $y = y(x,\varepsilon)$ for each sufficiently small ε. This problem exhibits two layers at x = a and x = b.

Using quasilinarization equation (3) can be written in the form

$$-\varepsilon^2 y'' + b(x)y(x) = f(x); \qquad x \in [0,1] \qquad (5)$$

with boundary conditions

$y(0) = \alpha$ and $y(1) = \beta$

Assume *b(x)* > 0 for $x \in [0,1]$. This problem exhibits two layers at x = 0 and x = 1.

The solution of the reduced problem is

$$y_0(x) = \frac{f(x)}{b(x)} \qquad (6)$$

which does not satisfy both the boundary conditions. The solution of (5) will be of the form

$$y(x) = y_0 + v_0 + w_0 \qquad (7)$$

where v_0 is the left boundary layer function (or solution) and w_0 is the right boundary layer function (or solution).

v_0, w_0 satisfy the differential equations

$$\frac{-d^2 v_0(\tau)}{d\tau^2} + b(0)v_0(\tau) = 0; \qquad \tau \in (0,\infty) \quad (8)$$

$$\frac{-d^2 w_0(\eta)}{d\eta^2} + b(1)w_0(\eta) = 0; \qquad \eta \in (0,\infty) \quad (9)$$

with

$$\begin{aligned} &v_0(\tau = 0) + w_0(\eta = 1/\varepsilon) = \alpha - y_0(0) \\ &v_0(\tau = 1/\varepsilon) + w_0(\eta = 0) = \beta - y_0(1) \\ &v_0(\tau = \infty) = w_0(\eta = \infty) = 0 \end{aligned}$$

Where $\tau = x/\varepsilon$ and $\eta = (1-x)/\varepsilon$

Solutions of (8) and (9) are given by

$$v_0(\tau) = Ae^{-\sqrt{b(0)}\tau} \tag{10}$$

$$w_0(\eta) = Be^{-\sqrt{b(1)}\eta} \tag{11}$$

Therefore, solution of (5) becomes

$$y(x) = y_0(x) + Ae^{-\frac{\sqrt{b(0)}}{\varepsilon}x} + Be^{-\frac{\sqrt{b(1)}}{\varepsilon}(1-x)} \tag{12}$$

where A and B are given by

$$A = \frac{(\beta - y_0(1)) - (\alpha - y_0(0))e^{-\frac{\sqrt{b(0)}}{\varepsilon}}}{1 - e^{-\frac{(\sqrt{b(0)}+\sqrt{b(1)})}{\varepsilon}}} \tag{13}$$

$$B = \frac{(\alpha - y_0(0)) - (\beta - y_0(1))e^{-\frac{\sqrt{b(1)}}{\varepsilon}}}{1 - e^{-\frac{(\sqrt{b(0)}+\sqrt{b(1)})}{\varepsilon}}} \tag{14}$$

Now we describe the fitted Numerov method for solving the differential equations of the form (5) as follows:

We rewrite the differential equation

$-\varepsilon^2 y'' + b(x)y(x) = f(x)$ as

$\varepsilon^2 y''(x) = g(x, y)$ where $g(x, y) = b(x)y(x) - f(x)$

We divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, x_2, \ldots\ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih\,; i = 0,1,\ldots..N$. We choose n such that $x_n = \frac{1}{2}$. In the interval $\left[0, \frac{1}{2}\right]$ the boundary layer will be in the left hand side i.e., at $x = 0$ and in the interval $\left[\frac{1}{2}, 1\right]$ the boundary layer will be in the right hand side i.e., at $x = 1$.

At $x = x_i$ the above differential equation can be written as

$\varepsilon^2 y_i''(x) = g(x_i, y_i)$ where $g(x_i, y_i) = b(x_i)y(x_i) - f(x_i)$

By Numerov method, we have

$$\varepsilon^2\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{12}(g_{i-1} + 10g_i + g_{i+1})$$

$$\varepsilon^2\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) = \frac{1}{12}\left(b_{i-1}y_{i-1} - f_{i-1} + 10\,b_i y_i - 10f_i + b_{i+1}y_{i+1} - f_{i+1}\right)$$

$$\varepsilon^2\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{12}\left(b_{i-1}y_{i-1} + 10b_i y_i + b_{i+1}y_{i+1}\right) = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right) \tag{15}$$

In the interval $\left[0, \frac{1}{2}\right]$, we introduce a fitting factor σ in the above difference scheme since the boundary layer is at $x = 0$ as:

$$\varepsilon^2\sigma\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{12}\left(b_{i-1}y_{i-1} + 10b_i y_i + b_{i+1}y_{i+1}\right) = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right) \tag{16}$$

for $i = 1,2,\ldots,n-1$

To find σ on the left boundary layer we use the asymptotic solution

$$v_0(x_i) = y_i = Ae^{-\frac{\sqrt{b(0)}}{\varepsilon}x_i} \tag{17}$$

and A is given by (13). We assume that solution converges uniformly to the solution of (5), then $f_{i-1} + 10f_i + f_{i+1}$ is bounded.

As $h \to 0$ equation (12) becomes

$$\lim_{h\to 0}\frac{\sigma}{\rho^2}(y_{i-1} - 2y_i + y_{i+1}) = \frac{b(0)}{12}\lim_{h\to 0}(y_{i-1} + 10y_i + y_{i+1}) \tag{18}$$

where $\rho = \frac{h}{\varepsilon}$ Substituting (17) in (18) and simplifying, we get the fitting factor as

$$\sigma = \frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 10\right)}{48\,Sinh^2\left(\frac{\sqrt{b(0)}\rho}{2}\right)} \tag{19}$$

which is a constant fitting factor.

Substituting the fitting factor (19) in (16), we have the three term recurrence relation as

$L_h[y_i] = H_i$, for $i = 1,2,\ldots,n-1$ (20)

where the difference operator $L_h[y_i]$ given by $L_h[y_i] = E_i y_{i-1} - F_i y_i + G_i y_{i+1}$.

Here

$$E_i = \frac{\varepsilon^2\sigma}{h^2} - \frac{b_{i-1}}{12}$$

$$F_i = \frac{2\varepsilon^2\sigma}{h^2} + \frac{10}{12}b_i$$

$$G_i = \frac{\varepsilon^2\sigma}{h^2} - \frac{b_{i+1}}{12}$$

$$H_i = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right)$$

Similarly, for the boundary layer at the right hand side, i.e., at x=1. We introduce a fitting factor σ_1 in the difference scheme (15) as

$$\varepsilon^2\sigma_1\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{12}\left(b_{i-1}y_{i-1} + 10b_iy_i + b_{i+1}y_{i+1}\right) = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right) \quad (21)$$

for $i = n+1, n+2, ...N-1$.

To find σ_1 on the right boundary layer we use the asymptotic solution

$$w_0(x_i) = y_i = Be^{-\frac{\sqrt{b(1)}}{\varepsilon}(1-x_i)} \quad (22)$$

where B is given by (14). Assume that solution converges uniformly to the solution of (5), then $f_{i-1} + 10f_i + f_{i+1}$ is bounded.

As $h \to 0$ equation (21) becomes

$$\lim_{h\to 0} \frac{\sigma_1}{\rho^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = \frac{b(1)}{12} \lim_{h\to 0}\left(y_{i-1} + 10y_i + y_{i+1}\right) \quad (23)$$

where $\rho = \dfrac{h}{\varepsilon}$

Substituting (22) in (23) and simplifying, we get the fitting factor as

$$\sigma_1 = \frac{\rho^2 b(1)\left(e^{\sqrt{b(1)}\rho} + e^{-\sqrt{b(1)}\rho} + 10\right)}{48\,Sinh^2\left(\frac{\sqrt{b(1)}\rho}{2}\right)} \quad (24)$$

which is a constant fitting factor.

From (21), we have the three term recurrence relation as

$$L_h[y_i] = H_i, \text{ for } i = n+1, n+2, \ldots N-1 \quad (25)$$

Here the difference operator $L_h[y_i]$ given by $L_h[y_i] = E_i y_{i-1} - F_i y_i + G_i y_{i+1}$

Where

$$E_i = \frac{\varepsilon^2\sigma_1}{h^2} - \frac{b_{i-1}}{12}$$

$$F_i = \frac{2\varepsilon^2\sigma_1}{h^2} + \frac{10}{12}b_i$$

$$G_i = \frac{\varepsilon^2\sigma_1}{h^2} - \frac{b_{i+1}}{12}$$

$$H_i = \frac{-1}{12}\left(f_{i-1} + 10f_i + f_{i+1}\right)$$

Note that the value of $y_n = y\left(x = \frac{1}{2}\right)$ is obtained by the solution of the reduced problem.

We solve the tridiagonal system given by (20) and (25) along with the value of $y_n = y\left(x = \frac{1}{2}\right)$ by Thomas algorithm.

Remark: When $b(0) = b(1)$, both the fitting factors become equal and the constant fitting factor is

$$\sigma = \frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho} + e^{-\sqrt{b(0)}\rho} + 10\right)}{48Sinh^2\left(\frac{\sqrt{b(0)}\rho}{2}\right)}$$

Since $b(x) > 0$, the difference operator L_h in (20) and (25) is positive type and hence there exists a unique solution for each set of given data and for each $\varepsilon > 0, h > 0$. That is, the difference operator L_h of the form $L_h[y_i] = E_i y_{i-1} - F_i y_i + G_i y_{i+1}$ satisfies the following:

(i) $E_i > 0,\ G_i > 0$ for all i, and

(ii) $E_i - F_i + G_i < 0$ for all i.

With this restriction on $b(x)$, L_h satisfies a discrete maximum principle.

3. Truncation error

From the finite differences, we have

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = y_i'' + \frac{h^2}{12}y_i^{(4)} + \frac{h^4}{360}y_i^{(6)} \quad (26)$$

and $y''_{i-1} - 2y''_i + y''_{i+1} = h^2 y_i^{(4)} + \frac{h^4}{12} y_i^{(6)}$ (27)

dividing equation (27) by 12, adding and subtracting y''_i to it, we get

$$y''_i + \frac{y''_{i-1} - 2y''_i + y''_{i+1}}{12} = y''_i + \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{144} y_i^{(6)} \quad (28)$$

From equations (26) and (28), we have

$$y''_i + \frac{y''_{i-1} - 2y''_i + y''_{i+1}}{12} - \frac{h^4}{144} y_i^{(6)} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \frac{h^4}{360} y_i^{(6)} \quad (29)$$

This equation can be written as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = \frac{1}{12}\left(g_{i-1} + 10g_i + g_{i+1}\right) - \frac{h^4}{240} y_i^{(6)} \quad (30)$$

which is a Numerov finite difference scheme for the differential equation $y'' = g(x, y)$.

The equation (28) can be written as

$$y''_i + \frac{\delta^2 y''_i}{12} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{h^4}{240} y_i^{(6)}$$

$$y''_i = \frac{\delta^2 y_i}{h^2\left(1 + \frac{\delta^2}{12}\right)} + \frac{h^4}{240} y_i^{(6)} \quad (31)$$

If we substitute (31) in differential equation $\varepsilon^2 y'' = g(x, y)$, we have the difference scheme

$$\varepsilon^2 \sigma\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) - \frac{1}{12}\left(b_{i-1} y_{i-1} + 10 b_i y_i + b_{i+1} y_{i+1}\right) = \frac{-1}{12}\left(f_{i-1} + 10 f_i + f_{i+1}\right) - \frac{\varepsilon^2 \sigma h^4}{240} y_i^{(6)}$$

Here $\delta^2 y_i = y_{i-1} - 2y_i + y_{i+1}$.

Hence the fitted Numerov method for $\varepsilon y'' = g(x, y)$ is $\varepsilon\sigma \dfrac{\delta^2 y_i}{h^2\left(1 + \frac{\delta^2}{12}\right)} = g(x, y) - \dfrac{\sigma\varepsilon^2 h^4}{240} y_i^{(6)}$

and the truncation error in the method is $|\tau_i| \le \max\limits_{x_{i-1} \le x \le x_{i+1}} \left\{ \dfrac{\sigma\varepsilon^2 h^4}{240} \left| y^{(6)}(x) \right| \right\}$

4. Numerical experiments

In this section we present two linear and one non-linear singularly perturbed third order boundary value problems to illustrate the method described in this paper. We presented the relative errors with L_2-norm and compared with the relative errors with L_2-norm by the classical finite difference method to support the method for different values of ε.

Example 1. Consider the boundary value problem [9]

$\varepsilon^2 y''' = y' - xy$ with $y(0) = y'(0) = y(1) = 1$

The uniform asymptotic solution of the problem is given by

$$y(x) = e^{\frac{x^2}{2}}\left[1 + \varepsilon + \varepsilon^2\left(\frac{x^4}{4} + \frac{3x^2}{2}\right)\right] - \varepsilon e^{-\frac{x}{\varepsilon}} +$$

$$e^{\frac{-(1-x)}{\varepsilon}}\left[\left(\sqrt{e} - 1\right)\left(\frac{x^2}{8} + \frac{x}{4} - \frac{11}{8}\right) + \varepsilon\left(-9\sqrt{e} - 3 + 3x + x\sqrt{e}\right)/8 - \frac{7\varepsilon^2\sqrt{e}}{4}\right]$$

The relative errors with L_2-norm are presented in table 1.

Example 2. Consider the boundary value problem

$\varepsilon^2 y''' = m^2 y' - y$ with $y(0)=1, y'(0)=1, y(1)=1$ with m = 2

The asymptotic solution of the problem is given by [4]

$$y(x,\varepsilon) \sim y_0(x) + O\left(\varepsilon |y_0'(0)| e^{-\frac{mx}{\varepsilon}}\right) + O\left(|y_0(1)-1| e^{-\frac{m(1-x)}{\varepsilon}}\right) + O(\varepsilon^2)$$

where $y_0(x) = e^{\frac{x}{m^2}}$ is the solution of the reduced problem.

The relative errors with L_2-norm are presented in table 2 and the relative errors with L_2-norm by classical finite difference method are presented table 3 for comparison.

Example 3. Consider the non-linear boundary value problem

$\varepsilon^2 y''' = 2yy'$ with $y(0)=1, y'(0)=1, y(1)=2$

The solution of the problem is given by [5]

$$y(x,\varepsilon) \sim 1 + O\left(e^{-\frac{\gamma(1-x)}{\varepsilon}}\right) \quad \text{for a constant } \gamma \text{ in } \left(0, \sqrt{2}\right)$$

The relative errors with L_2-norm are presented in table 4.

Table 1. The relative errors with L_2-norm of example (1)

ε^2 \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	7.92(-2)	8.19(-2)	8.07(-2)	7.99(-2)	7.95(-2)	7.92(-2)	7.91(-2)	7.90(-2)
2^{-4}	1.03(-1)	1.05(-1)	1.05(-1)	1.04(-1)	1.04(-1)	1.04(-1)	1.04(-1)	1.04(-1)
2^{-5}	9.86(-2)	1.01(-1)	1.01(-1)	1.01(-1)	1.01(-1)	1.01(-1)	1.01(-1)	1.01(-1)
2^{-8}	4.91(-2)	5.15(-2)	5.22(-2)	5.24(-2)	5.24(-2)	5.25(-2)	5.25(-2)	5.25(-2)
2^{-10}	2.61(-2)	2.78(-2)	2.84(-2)	2.86(-2)	2.86(-2)	2.87(-2)	2.87(-2)	2.87(-2)
2^{-20}	2.80(-3)	6.89(-4)	8.46(-4)	9.37(-4)	9.63(-4)	9.71(-4)	9.73(-4)	9.73(-4)

Table 2. The relative errors with L_2-norm of example (2) for *m* = 2

ε^2 \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	5.83(-2)	4.63(-2)	4.20(-2)	4.02(-2)	3.94(-2)	3.90(-2)	3.88(-2)	3.87(-2)
2^{-4}	4.19(-2)	2.93(-2)	2.51(-2)	2.35(-2)	2.28(-2)	2.25(-2)	2.23(-2)	2.22(-2)
2^{-5}	3.31(-2)	1.98(-2)	1.56(-2)	1.40(-2)	1.34(-2)	1.31(-2)	1.30(-2)	1.29(-2)
2^{-8}	2.24(-2)	1.01(-2)	5.10(-3)	3.50(-3)	3.00(-3)	2.80(-3)	2.70(-3)	2.70(-3)
2^{-10}	1.66(-2)	8.00(-3)	3.60(-3)	3.00(-3)	1.20(-3)	1.10(-3)	9.44(-4)	9.44(-4)

2^{-20}	9.80(-3)	3.60(-3)	1.30(-3)	5.07(-4)	2.08(-4)	9.35(-5)	4.45(-5)	1.98(-5)

Table 3. The relative errors with L_2-norm of example (2) for *m* = 2 by classical finite difference scheme

ε^2 \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	1.42(-1)	1.37(-1)	1.38(-1)	1.39(-1)	1.41(-1)	1.42(-1)	1.42(-1)	1.42(-1)
2^{-4}	9.27(-2)	8.96(-2)	9.08(-2)	9.28(-2)	9.43(-2)	9.53(-2)	9.59(-2)	9.62(-2)
2^{-5}	6.18(-2)	5.90(-2)	6.08(-2)	6.30(-2)	6.47(-2)	6.58(-2)	6.63(-2)	6.66(-2)
2^{-8}	2.58(-2)	1.69(-2)	1.74(-2)	1.94(-2)	2.11(-2)	2.21(-2)	2.27(-2)	2.31(-2)
2^{-10}	2.14(-2)	9.50(-3)	7.20(-3)	8.20(-3)	9.50(-3)	1.05(-2)	1.10(-2)	1.14(-2)
2^{-20}	2.00(-2)	7.00(-3)	2.40(-3)	8.64(-4)	3.21(-4)	2.62(-4)	4.27(-4)	3.41(-4)

Table 4. The relative errors with L_2-norm of example (3)

ε^2 \ h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
2^{-3}	4.47(-2)	3.96(-2)	3.76(-2)	3.68(-2)	3.64(-2)	3.62(-2)	3.62(-2)	3.61(-2)
2^{-4}	2.85(-2)	2.30(-2)	2.10(-2)	2.03(-2)	1.99(-2)	1.97(-2)	1.97(-2)	1.96(-2)
2^{-5}	2.13(-2)	1.51(-2)	1.30(-2)	1.22(-2)	1.19(-2)	1.17(-2)	1.16(-2)	1.16(-2)
2^{-8}	1.38(-2)	6.90(-3)	4.10(-3)	3.20(-3)	2.80(-3)	2.70(-3)	2.60(-3)	2.60(-3)
2^{-10}	8.80(-3)	5.30(-3)	2.60(-3)	1.50(-3)	1.20(-3)	1.00(-3)	9.79(-4)	9.57(-4)
2^{-20}	2.81(-4)	2.18(-4)	1.62(-4)	1.18(-4)	8.49(-5)	5.88(-5)	3.28(-5)	1.51(-5)

5. Conclusions

In this paper, we presented a asymptotic-numerical method for a class of third order singularly perturbed boundary value problems with suitable boundary conditions. The third order boundary value problem is transformed to asymptotically equivalent second order boundary value problem. This problem is solved efficiently by using fitted Numerov method. Two linear and one non-linear example are solved to illustrate the method and relative errors with L_2-norms are presented to support the method. To show the efficiency of the method we compare results of one of example by the classical finite difference method for the given third order singular perturbation problem.

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