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Numerical Integration Method for Singularly Perturbed Delay Differential Equations

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Abstract: In this paper, we present a numerical integration method to solve singularly perturbed delay differential equations. In this method, we first convert the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employ the numerical integration. Then, linear interpolation is used to get three term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing several model examples by taking various values for the delay parameter and perturbation parameter.

Keywords: Singularly perturbed boundary value problems; delay term; boundary layer; integration method.

1. Introduction

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term. In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of processes in various application fields, for e.g., the first exit time problem in the modeling of the activation of neuronal variability [11], in the study of bistable devices [2], and variational problems in control theory [6] where they provide the best and in many cases the only realistic simulation of the observed. Lange and Miura [11, 12] gave an asymptotic approach for a class of boundary-value problems for linear second-order singularly perturbed differential-difference equations.

Kadalbajoo and Sharma [9, 10], presented a numerical approaches to solve singularly perturbed differential-difference equation, which contains negative shift in the derivative term or in the function but not in the derivative term.

In [5], the authors Gabil M. Amiraliyev, Erkan Cimen had given a numerical method for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. The authors presented an exponentially fitted difference scheme on a uniform mesh which is accomplished by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. In [7], the authors Jugal Mohapatra, Srinivasan Natesan constructed a numerical method for a class of singularly perturbed differential-differen-

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ce equations with small delay. The numerical method comprises of upwind finite difference operator on an adaptive grid, which is formed by equidistributing the arc-length monitor function. In[13], the authors M.K. Kadalbajoo, Devendra Kumar presented a numerical method for singularly perturbed boundary value problem for a linear second order differential-difference equation of the convection-diffusion type with small delay parameter. Taylor series is used to tackle the delay term. The fitted mesh technique is employed to generate a piecewise-uniform mesh, condensed in the neighbourhood of the boundary layers. B-spline collocation method is used with fitted mesh.

There are wide varieties of asymptotic expansion methods for solving singular perturbation problems. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in inner and outer regions, which are not routine exercise but require skill, insight and experimentation. Even the matching of the coefficients of the inner and outer solution expansions can be a demanding process. Hence, we require the other ways to attack singular perturbation problems; ways that are very easy to use and ready for computer implementation.

In this paper, we present a numerical integration method to solve singularly perturbed delay differential equations. In this method, we first convert the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employ the numerical integration. Then, linear interpolation is used to get three term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing several model examples by taking various values for the delay parameter and perturbation parameter.

2. Description of the method

2.1. Layer on the left side

Consider singularly perturbed delay differential equation of the form

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

with boundary conditions

$$y(0) = \alpha, \; -\delta \le x \le 0 \tag{2a}$$

and

$$y(1) = \beta \tag{2b}$$

where ε is small parameter, $0 < \varepsilon << 1$ and δ is also small shifting parameter, $0 < \delta << 1$; *a(x), b(x), f(x)* are bounded continuous functions in (0, 1) and α, β are finite constants. Further, we assume that $a(x) \ge M > 0$ throughout the interval [0, 1], where M is positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

By using Taylor series expansion in the neighbourhood of the point *x,* we have

$$y'(x-\varepsilon) \approx y'(x) - \varepsilon y''(x) \tag{3}$$

and consequently, equation (1) is replaced by the following approximate first order differential equation with a small deviation argument:

$$y'(x) = y'(x-\varepsilon) - a(x)y'(x-\delta) - b(x)y + f(x) \tag{4}$$

The transition from equation (1) to equation (4) is admitted, because of the condition that ε is small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Elsgolt's and Norkin [4].

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h=1/N$ so that $x_i = ih,\ i = 0, 1, 2, \ldots, N.$

Integrating eq. (4) with respect to x from x_i to x_{i+1}, we get

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'(x-\varepsilon)dx + \int_{x_i}^{x_{i+1}} a(x)y'(x-\delta)dx - \int_{x_i}^{x_{i+1}} b(x)ydx + \int_{x_i}^{x_{i+1}} f(x)dx$$

$$y_{i+1} - y_i = y(x_{i+1}-\varepsilon) - y(x_i-\varepsilon) - [a_{i+1}y(x_{i+1}-\delta) - a_i y(x_i-\delta)]$$
$$+ \int_{x_i}^{x_{i+1}} (a'y(x-\delta) - by + f)dx$$

By using Trapezoidal rule to evaluate the integral, we get

$$y_{i+1} - y_i = y(x_{i+1}-\varepsilon) - y(x_i-\varepsilon) - [a_{i+1}y(x_{i+1}-\delta) - a_i y(x_i-\delta)]$$
$$+\frac{h}{2}\left(a_i'y(x_i-\delta) + a_{i+1}'y(x_{i+1}-\delta) - b_i y_i - b_{i+1}y_{i+1} + f_i + f_{i+1}\right) \tag{5}$$

Again, by means of Taylor series expansion and then by approximating $y'(x)$ by linear interpolation, we get,

$$y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1}$$

$$y(x_{i+1}-\delta) \approx y(x_{i+1}) - \delta\, y'(x_{i+1}) = y_{i+1} - \delta\left(\frac{y_{i+1} - y_i}{h}\right) = \left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i$$

By making use of the above equations in (5) we obtain

$$\varepsilon y_{i+1}' - \varepsilon y_i' = (1-\frac{\delta}{h})(\frac{h}{2}a_{i+1}' - a_{i+1})y_{i+1} + \frac{\delta}{h}(\frac{h}{2}a_{i+1}' - a_{i+1})y_i + (1-\frac{\delta}{h})(a_i + \frac{h}{2}a_i')y_i$$
$$+ (a_i + \frac{h}{2}a_i')\frac{\delta}{h}y_{i-1} - \frac{h}{2}a_i'y_i - \frac{h}{2}a_{i+1}'y_{i+1} + \frac{h}{2}(f_i + f_{i+1})$$

$$\varepsilon\frac{(y_{i+1}-y_i)}{h} - \varepsilon\left(\frac{y_i - y_{i-1}}{h}\right) = \left[\left(1-\frac{\delta}{h}\right)\left(\frac{h}{2}a_{i+1}' - a_{i+1}\right) - \frac{h}{2}a_{i+1}\right]y_{i+1}$$
$$+\left(\frac{\delta}{h}\left(\frac{h}{2}a_{i+1}' - a_{i+1}\right) + (1-\frac{\delta}{h})(a_i + \frac{h}{2}a_i') - \frac{h}{2}a_i\right)y_i$$
$$+ (a_i + \frac{h}{2}a_i')\frac{\delta}{h}y_{i-1} + \frac{h}{2}(f_i + f_{i+1})$$

Rearranging the above equation as three term recurrence relation, we get

$$\left[\frac{\varepsilon}{h} - \left(a_i + \frac{ha_i'}{2}\right)\frac{\delta}{h}\right]y_{i-1} - \left[\frac{2\varepsilon}{h} + \frac{\delta}{h}\left(\frac{h}{2}a_{i+1}' - a_{i+1}\right) + (1-\frac{\delta}{h})(a_i + \frac{h}{2}a_i') - \frac{h}{2}b_i\right]y_i$$
$$+\left[\frac{\varepsilon}{h} - (1-\frac{\delta}{h})\left(\frac{h}{2}a_{i+1}' - a_{i+1}\right) - \frac{h}{2}b_{i+1}\right]y_{i+1} = \frac{h}{2}(f_i + f_{i+1})$$

which can be written as

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i$, for i = 1, 2, …, n-1. (6)

where

$$E_i = \frac{\varepsilon}{h} - (a_i + \frac{h}{2}a_i')\frac{\delta}{h}, \quad F_i = \frac{2\varepsilon}{h} + (a_i + \frac{h}{2}a_i')(1-\frac{\delta}{h}) + \frac{\delta}{h}(-a_{i+1} + \frac{h}{2}a_{i+1}') - \frac{h}{2}b_i$$

$$G_i = \frac{\varepsilon}{h} - (-a_{i+1} + \frac{h}{2}a_{i+1}')(1-\frac{\delta}{h}) - \frac{h}{2}b_{i+1}, \quad H_i = \frac{h}{2}(f_i + f_{i+1})$$

Equation (6) is a tridiagonal system and we solve it by using method of discrete invariant imbedding.

2.2. Right - end boundary layer problem

We now assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 1$.

By using Taylor series expansion in the neighbourhood of the point x, we have

$$y'(x+\varepsilon) \approx y'(x) + \varepsilon y''(x) \tag{7}$$

and consequently, equation (1) is replaced by the following approximate first order differential equation with a small deviation argument:

$$y'(x) = y'(x+\varepsilon) + a(x)y'(x-\delta) + b(x)y - f(x) \tag{8}$$

Now we divide the interval [0, 1] into N equal subintervals of mesh size h=1/N so that $x_i = ih,\ i = 0, 1, 2, \ldots, N.$

Integrating eq. (8) with respect to x from x_{i-1} to x_i, we get

$$y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} y'(x+\varepsilon)dx + \int_{x_{i-1}}^{x_i} a(x)y'(x-\delta)dx + \int_{x_{i-1}}^{x_i} b(x)y dx - \int_{x_{i-1}}^{x_i} f(x)dx$$

$$y_i - y_{i-1} = y(x_i+\varepsilon) - y(x_{i-1}+\varepsilon) + [a_i y(x_i-\delta) - a_{i-1}y(x_{i-1}-\delta)] + \int_{x_{i-1}}^{x_i} (-a'y(x-\delta) + by - f)dx$$

By using Trapezoidal rule to evaluate the integral, we get

$$y_i - y_{i-1} = y(x_i+\varepsilon) - y(x_{i-1}+\varepsilon) + [a_i y(x_i-\delta) - a_{i-1}y(x_{i-1}-\delta)] + \frac{h}{2}\left(-a_{i-1}'y(x_{i-1}-\delta) - a_i'y(x_i-\delta) + b_{i-1}y_{i-1} + b_i y_i - f_{i-1} - f_i\right) \tag{9}$$

Again, by means of Taylor series expansion and then by approximating $y'(x)$ by linear interpolation, we get,

$$y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_{i+1}-y_i}{h}\right) = \left(1+\frac{\delta}{h}\right)y_i - \frac{\delta}{h}y_{i+1}$$

$$y(x_{i-1}-\delta) \approx y(x_{i-1}) - \delta\, y'(x_{i-1}) = y_{i-1} - \delta\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1+\frac{\delta}{h}\right)y_{i-1} - \frac{\delta}{h}y_i$$

By making use of the above equations in (9) we obtain

$$\varepsilon y'_{i-1} - \varepsilon y'_i = (a_i - \frac{h}{2}a'_i)y(x_i - \delta) - (a_{i-1} + \frac{h}{2}a'_{i-1})y(x_{i-1} - \delta) + \frac{h}{2}b_{i-1}y_{i-1} + \frac{h}{2}b_i y_i - \frac{h}{2}(f_i + f_{i+1})$$

$$\varepsilon\left[\frac{(y_i - y_{i-1})}{h} - \left(\frac{y_{i+1} - y_i}{h}\right)\right] = (a_i - \frac{h}{2}a'_i)\left[y_i - \delta\left(\frac{y_{i+1} - y_i}{h}\right)\right] - \left(a_{i-1} + \frac{h}{2}a'_{i-1}\right)\left[y_{i-1} - \delta\left(\frac{y_i - y_{i-1}}{h}\right)\right] + \frac{h}{2}b_{i-1}y_{i-1} + \frac{h}{2}b_i y_i - \frac{h}{2}(f_i + f_{i-1})$$

Rearranging the above equation as three term recurrence relation, we get

$$\left[\frac{-\varepsilon}{h} + (a_{i-1} + \frac{h}{2}a'_{i-1})(1 + \frac{\delta}{h}) - \frac{h}{2}b_{i-1}\right]y_{i-1}$$

$$-\left[\frac{-2\varepsilon}{h} + (a_i - \frac{h}{2}a'_i)(1 + \frac{\delta}{h}) + \frac{\delta}{h}(a_{i-1} + \frac{h}{2}a'_{i-1}) + \frac{h}{2}b_i\right]y_i$$

$$+\left[\frac{-\varepsilon}{h} + (a_i - \frac{h}{2}a'_i)\frac{\delta}{h}\right]y_{i+1} = \frac{-h}{2}(f_i + f_{i-1})$$

which can be written as

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i$, for i = 1, 2, …, n-1. (10)

where

$$E_i = \frac{-\varepsilon}{h} + (a_{i-1} + \frac{h}{2}a'_{i-1})(1 + \frac{\delta}{h}) - \frac{h}{2}b_{i-1}$$

$$F_i = \frac{-2\varepsilon}{h} + (a_i - \frac{h}{2}a'_i)(1 + \frac{\delta}{h}) + \frac{\delta}{h}(a_{i-1} + \frac{h}{2}a'_{i-1}) + \frac{h}{2}b_i, \quad G_i = \frac{-\varepsilon}{h} + (a_i - \frac{h}{2}a'_i)\frac{\delta}{h}$$

$$H_i = \frac{-h}{2}(f_i + f_{i-1})$$

Equation (10) is a tridiagonal system and we solve it by using method of discrete invariant imbedding.

3. Discrete invariant imbedding algorithm

We now describe the Thomas algorithm which is also called Discrete Invariant Imbedding Angel & Bellman [1] to solve the three term recurrence relation:

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i$,

for i = 1, 2, … , n-1 (11)

Let us set a difference relation of the form

$y_i = W_i y_{i+1} + T_i$

for i = N-1, N-2, … , 2, 1 (12)

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (12), we have

$y_{i-1} = W_{i-1}y_i + T_{i-1}$ (13)

substituting (13) in (11), we have

$$E_i(W_{i-1}y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i$$

$$y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \quad (14)$$

By comparing (12) and (14), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) \quad (15)$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \quad (16)$$

To solve these recurrence relations for i = 2,3, … ,N-1, we need the initial conditions for W_0 and T_0. If we choose $W_0 = 0$, then we get $T_0 = \phi_0$. With these initial values, we compute W_i and T_i for i = 2, 3, … , N-1 from equations (15) and(16) in forward process, and then obtain y_i in the backward process from (12).

The conditions for the discrete invariant imbedding algorithm to be stable are, see [1, 8]:

$$E_i > 0, \quad G_i > 0, \quad F_i \geq E_i + G_i$$

$$\text{and} \quad |E_i| \leq |G_i| \quad (17)$$

In our method, one can easily show that if the assumptions *a(x)* > 0, *b(x)* < 0 and $(\varepsilon - \delta a(x)) > 0$ hold, then the above conditions (17) hold and thus the invariant imbedding algorithm is stable.

4. Numerical experiments

To demonstrate the efficiency of the method, we consider two numerical experiments with left-end boundary layer and two numerical experiments with right-end boundary layer. We compare the results with the exact solution of the problems. Also we have plotted the graphs of the exact and computed solution of the problem for different values of ε and for different values of δ of $o(\varepsilon)$, which are represented by solid and dotted lines respectively. The maximum absolute error for the examples not having the exact solution is calculated using the double mesh principle [3], $E^N = \max_{0 \leq i \leq N} \left| y_i^N - y_{2i}^{2N} \right|$.

Example 1. Consider an example of singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0; \quad x \in [0,1]$$

with $y(0) = 1$ and y(1) = 1.

Here *a(x)* = 1, *b(x)* = -1 and *f(x)* = 0.

The exact solution is given by

$$y(x) = \frac{((1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x})}{(e^{m_1} - e^{m_2})}$$

where $m_1 = (-1 - \sqrt{1 + 4(\varepsilon - \delta)}) \big/ 2(\varepsilon - \delta)$

and $m_2 = (-1 + \sqrt{1 + 4(\varepsilon - \delta)}) \big/ 2(\varepsilon - \delta)$.

From equation (4), the corresponding first order neutral type delay differential equation is $y'(x) = y'(x-\varepsilon) - y'(x-\delta) + y(x)$. We solve this problem by the present method and the maximum absolute errors are presented in Table1 and Table 2 for ε = 0.1, 0.01 and for different values of δ and compared with results of paper [9]. Also we have plotted the graphs of the exact and computed solution of the problem for ε = 0.1, 0.01 and for different values of δ as shown in Figure 1 and 2 respectively.

Example 2. Now we consider an example of variable coefficient singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0 \text{ with } y(0) = 1, \; y(1) = 1$$

Here $a(x) = e^{-0.5x}, b(x) = -1$ and $f(x) = 0$. The exact solution of the problem is not known. From equation (4), the corresponding first order neutral type delay differential equation is

$$y'(x) = y'(x-\varepsilon) - e^{-0.5x} y'(x-\delta) + y(x).$$

We solve this problem by the present method and The maximum absolute errors by the double mesh principle are presented in Table 3 for ε = 0.1 and for different values of δ.

Also we plot the graphs of the computed solution of the problem for $\varepsilon = 0.1, 0.01$ and for different values of δ as shown in Figure 3 and 4 respectively.

Example 3. Consider a singularly perturbed delay differential equation with right layer:
$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0$; $x \in [0,1]$
with $y(0) = 1$ and $y(1) = -1$.
Here $a(x) = -1$, $b(x) = -1$ and $f(x) = 0$.
The exact solution is given by
$$y(x) = \frac{((1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x})}{(e^{m_2} - e^{m_1})}$$
where $m_1 = {(1-\sqrt{1+4(\varepsilon+\delta)})}\big/{2(\varepsilon+\delta)}$
and $m_2 = {(1+\sqrt{1+4(\varepsilon+\delta)})}\big/{2(\varepsilon+\delta)}$.

From equation (8), the corresponding first order neutral type delay differential equation is $y'(x) = y'(x+\varepsilon) - y'(x-\delta) - y(x)$. We solve this problem by the present method and the maximum absolute errors are presented in Table 4 and Table 5 for $\varepsilon = 0.01, 0.001$ and for different values of δ and compared with results of paper [9]. Also we have plotted the graphs of the exact and computed solution of the problem for $\varepsilon = 0.1, 0.01$ and for different values of δ as shown in Figure 5 and 6 respectively.

Example 4. Now we consider an example of variable coefficient singularly perturbed delay differential equation with right layer:
$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0$, with $y(0) = 1$, $y(1) = 1$

Here $a(x) = e^x, b(x) = -x$ and $f(x) = 0$. The exact solution of the problem is not known. From equation (8), the corresponding first order neutral type delay differential equation is
$$y'(x) = y'(x+\varepsilon) - e^x y'(x-\delta) - y(x).$$
We solve this problem by the present method and the maximum absolute errors by the double mesh principle are presented in Table 6 for $\varepsilon = 0.1$ and for different values of δ. Also we plot the graphs of the computed solution of the problem for $\varepsilon = 0.1, 0.01$ and for different values of δ as shown in Figure 7 and 8 respectively.

5. Discussions and conclusions

We have presented a numerical integration method to solve singularly perturbed delay differential equations. In general numerical solution of second order differential equation will be more difficult than numerical solution of first order differential equation. In this method, we first converted the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employed the numerical integration, whereas in paper [9] the second order singularly perturbed delay differential equation is transformed to an approximate second order singular perturbation problem which is valid only for small values of delay parameter δ. Then, linear interpolation is used to get three term recurrence relation which is solved easily by method of invariant imbedding algorithm. The method is demonstrated by implementing several model examples by taking various values for the delay parameter an d perturbation parameter.

This method is very easy to implement. The effect of small shifts on the boundary layer solution of the problem has been given by considering several numerical experiments. It is observed that if $\delta = o(\varepsilon)$ and as δ increases, the thickness of the boundary layer decreases in the case when the solution exhibits layer behaviour on the left side, while in the case of the right side boundary layer, it increases and maximum error decreases as the grid size h decreases in both cases which shows the convergence to the computed solution.

Table 1. The maximum absolute errors of example 1 with ε = 0.1 for different values of δ and *h*

δ	N							
	10^2		10^3		10^4		10^5	
	present method	results in [9]	present method	results in [9]	present method	results in [9]	present method	results in [9]
0.01	0.01172	0.011824	0.00122	0.00122	1.231e-004	1.235e-004	1.228e-005	1.236e-005
0.03	0.01505	0.01515	0.00158	0.00159	1.598e-004	1.6020e-004	1.599e-005	1.603e-005
0.06	0.02575	0.02584	0.00281	0.00281	2.839e-004	2.842e-004	2.844e-005	2.845e-005
0.08	0.04781	0.083131	0.00562	0.01110	5.735e-004	1.151e-003	5.748e-005	5.748e-005

Table 2. The maximum absolute errors of example 1 with ε = 0.01 for different values of δ and h

δ	N							
	10^2		10^3		10^4		10^5	
	present method	results in [9]	present method	results in [9]	present method	results in [9]	present method	results in [9]
0.001	0.09073	0.09092	0.01228	0.01229	0.00127	0.00127	1.284e-004	1.284e-004
0.003	0.10803	0.10836	0.01562	0.01562	0.00164	0.00164	1.653e-004	1.653e-004
0.006	0.12777	0.12845	0.02630	0.02631	0.00287	0.00287	2.897e-004	2.897e-004
0.008	0.10040	0.10149	0.04833	0.04834	0.00568	0.00568	5.794e-004	5.794e-004

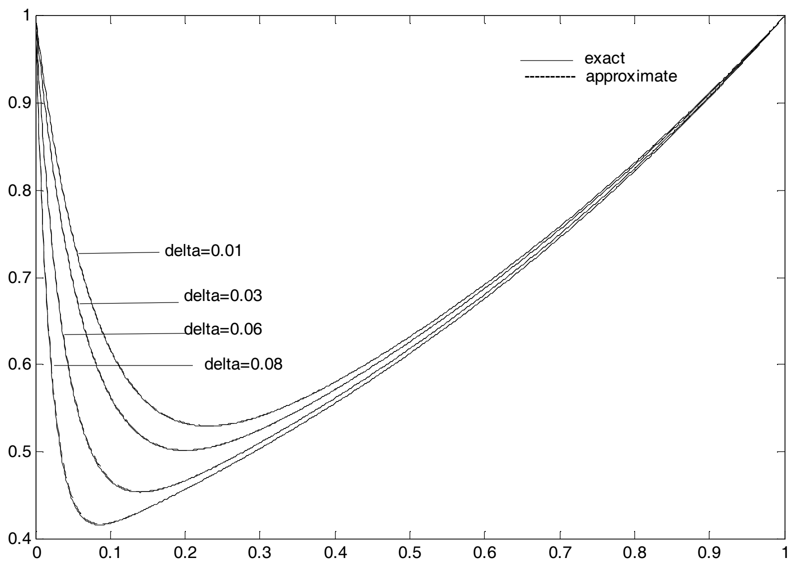

Figure 1. Graph of the solution of the example (1) for ε = 0.1 and for different δ of $o(\varepsilon)$

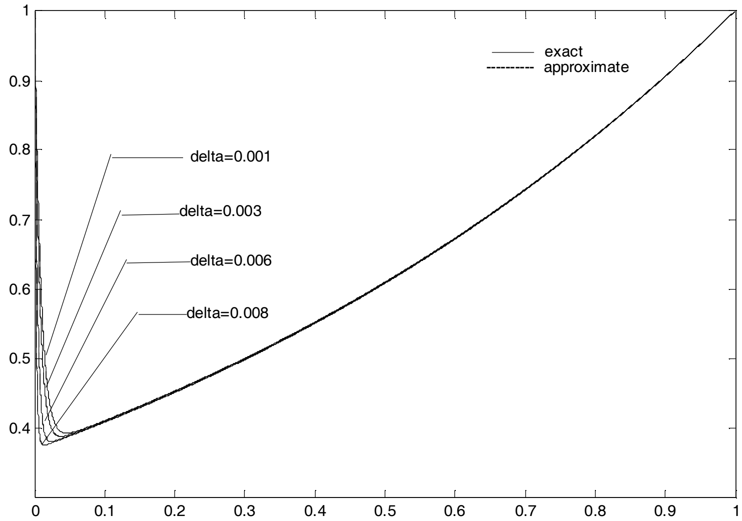

Figure 2. Graph of the solution of the example (1) for $\varepsilon = 0.01$ and for different δ of $o(\varepsilon)$

Table 3. The maximum absolute errors r of example 2 by double mesh principle with $\varepsilon = 0.1$ for different values of δ and grid size

δ	N		
	10^2	10^3	10^4
0.01	0.00632996	0.000674268	6.7871251e-005
0.03	0.00815917	0.000882563	8.8986856e-005
0.06	0.01384760	0.001579726	1.6020004e-004
0.08	0.02477158	0.003173235	3.2602775e-004

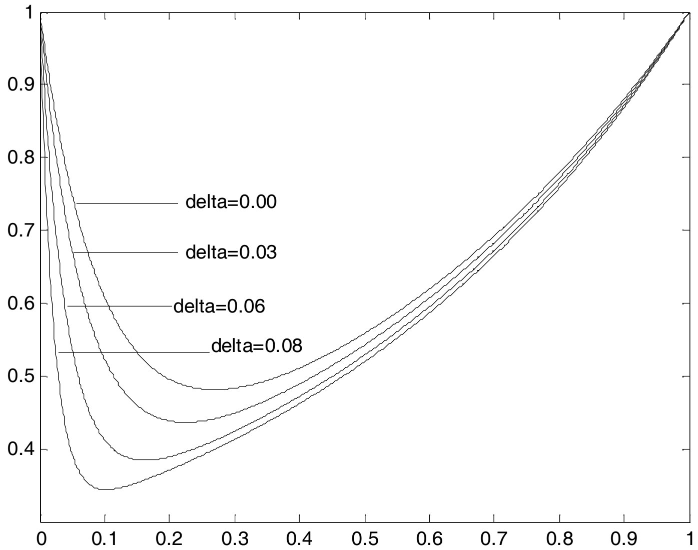

Figure 3. Graph of the solution of the example (2) for $\varepsilon = 0.1$ and for different δ of $o(\varepsilon)$

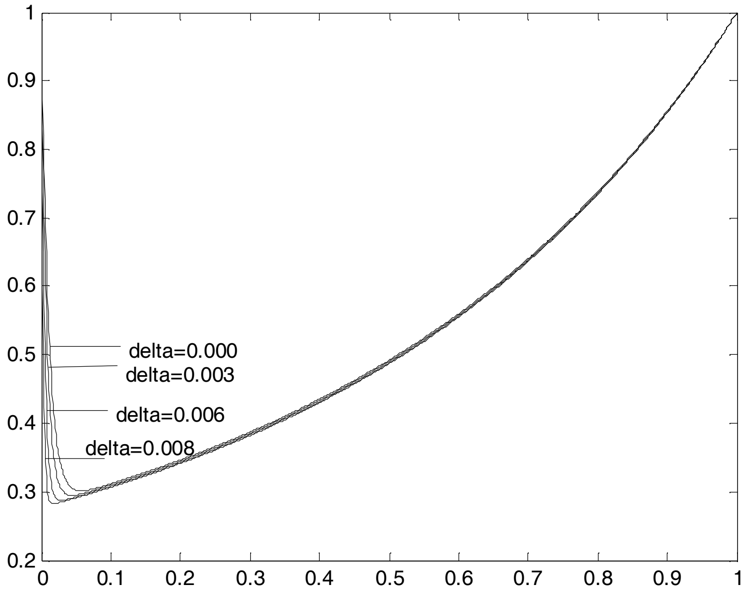

Figure 4. Graph of the solution of the example (2) for $\varepsilon = 0.01$ and for different δ of $o(\varepsilon)$

Table 4. The maximum absolute errors of example 3 with $\varepsilon = 0.01$ for different values of δ and h

δ	N							
	10^2		10^3		10^4		10^5	
	present method	results in [9]	present method	results in [9]	present method	results in [9]	present method	results in [9]
0.000	0.18113	0.17855	0.02422	0.02387	0.002512	0.00247	0.00025	0.00024
0.007	0.12064	0.11763	0.014515	0.01395	0.00148	0.00142	0.00014	0.00014
0.015	0.08667	0.08351	0.00996	0.00944	0.00101	0.00095	0.00009	0.00009
0.025	0.06466	0.06147	0.00717	0.00678	0.00072	0.00068	0.00007	0.00006

Table 5. The maximum absolute errors of example 3 with $\varepsilon = 0.001$ for different values of δ and h

δ	N							
	10^2		10^3		10^4		10^5	
	present method	results in [9]	present method	results in [9]	present method	results in [9]	present method	results in [9]
0.0007	0.21605	0.21339	0.02997	0.02897	0.00313	0.00301	0.000315	0.000302
0.0015	0.12615	0.12311	0.01539	0.01462	0.00157	0.00149	0.00015	0.00014
0.0025	0.08410	0.08096	0.00959	0.00911	0.00097	0.00092	0.00009	0.00009

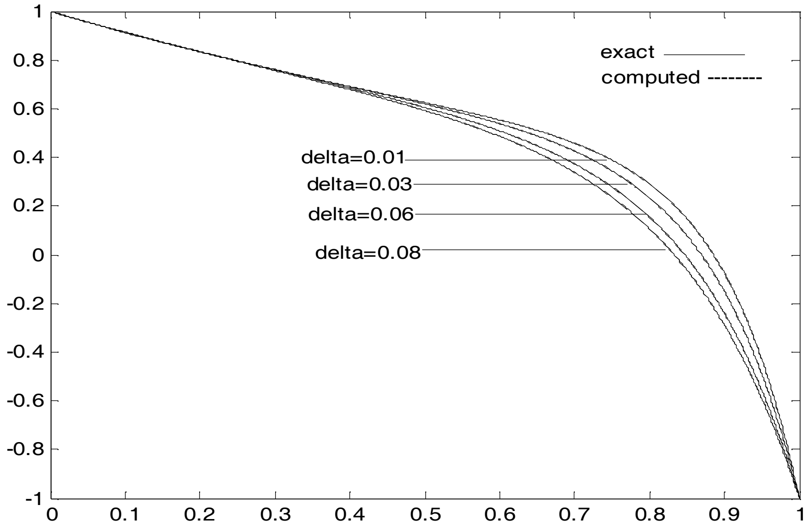

Figure 5. Graph of the solution of the example (3) for $\varepsilon = 0.1$ and for different δ of $o(\varepsilon)$

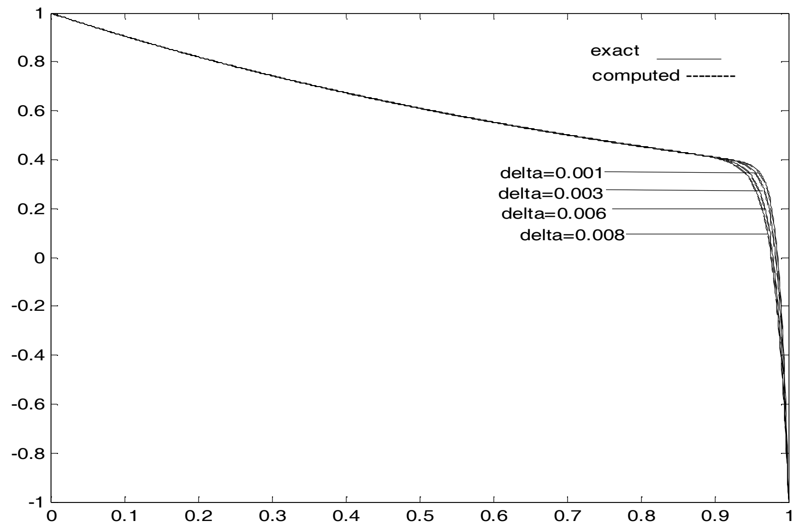

Figure 6. Graph of the solution of the example (3) for $\varepsilon = 0.01$ and for different δ of $o(\varepsilon)$

Table 6. The maximum absolute errors of example 4 by double mesh principle with $\varepsilon = 0.1$ for different values of δ and grid size h

δ	N		
	10^2	10^3	10^4
0.01	0.00575975	0.00050842	5.02478e-005
0.03	0.003932768	0.00036132	3.58384e-005
0.06	0.002702569	0.00025507	2.53643e-005
0.08	0.00224689	0.00021413	2.13134e-005

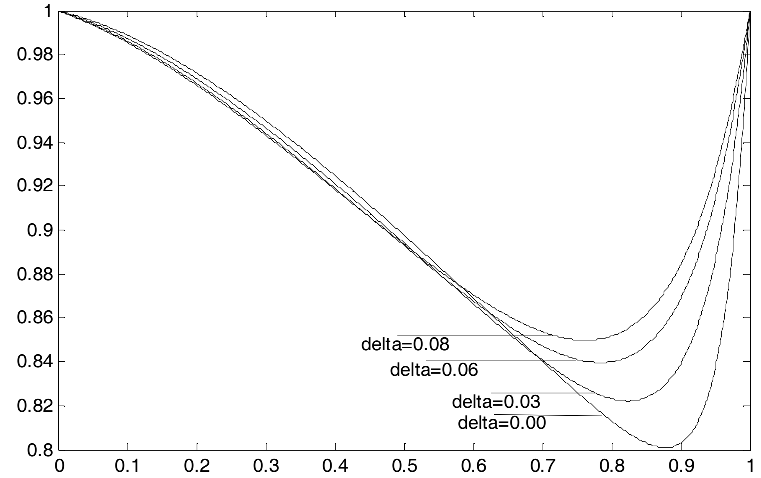

Figure 7. Graph of the solution of the example (4) for $\varepsilon = 0.1$ and for different δ of $o(\varepsilon)$

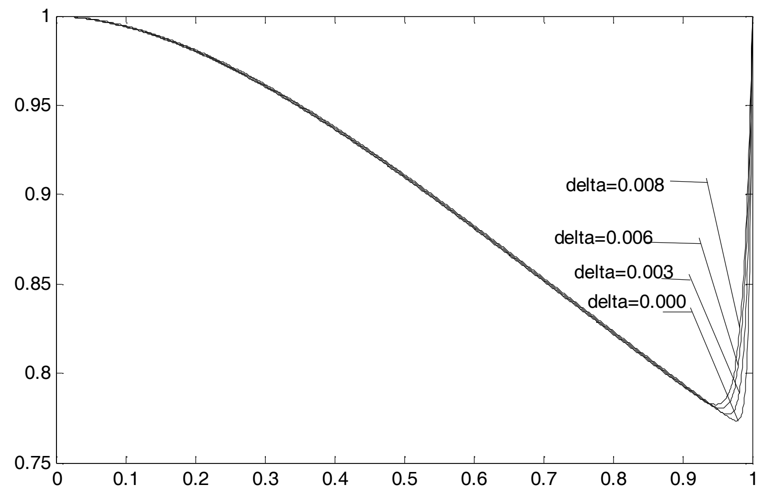

Figure 8. Graph of the solution of the example (4) for $\varepsilon = 0.01$ and for different δ of $o(\varepsilon)$

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