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A Numerical Patching Method for Solving Singular Perturbation Problems Via Padé Approximates

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Abstract: In this paper, we present a numerical patching method for solving a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. In order to know the behavior of the solution of the singular perturbation problem in the boundary layer region, it is always suggestive to solve the problem in outer and boundary layer regions separately. By constructing a modified problem with a boundary layer correction, the boundary layer is dealt with separately and series method used. The condition at infinity will be applied to the corresponding Padé approximates of the obtained series solution. Several problems are solved to demonstrate the applicability and efficiency of the proposed method. It is observed that the present method approximates the exact solution very well.

Keywords: Singular perturbation problems; boundary layer; boundary layer correction; Pade' approximates.

1. Introduction

Singularly perturbed second order two-point boundary value problems arise very frequently in fluid mechanics and other branches of Applied Mathematics. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. So, typically there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behaves regularly and vary slowly. The numerical treatment of the singular perturbation problems is far from the trivial because of the boundary layer behavior of the solution. There are a wide variety of methods for the solution of the singular perturbation problems. The notable methods are asymptotic expansion approximations. These methods consists of: (a) dividing the problem into an inner region (boundary layer) problem and an outer region problem; (b) expressing the inner and outer solutions as asymptotic expansions; (c) equating various terms in the inner and outer expressions to determine the constants in these expressions; and (d) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. Typically, the inner region problems are obtained from the original problem by rescaling the independent variable. These methods and their variations have been used successfully on a variety of linear and non-linear singular perturbation problems. However, there can be difficulties in applying these methods, such as the matching of the coefficients of the inner and outer expansions. Success may depend on finding the proper scaling or the proper transformation to express the dependent and independent

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variables. For a detailed theory and analytical discussion on singular perturbation problems one may refer to the books and high level monographs: O' Malley [1], Nayfeh [2], Kevorkian and Cole [3], Bender and Orszag [4]. For a detailed Numerical and Asymptotic discussion on singular perturbation problems one may refer to the books and high level monographs: Hemker [5], Hemker and Miller [6], Miller [7-12], Miller et.al. [13], Axelson et.al. [14], Doolan et.al. [15], Holmes [16], Miranker [17], Morton [18], Aiken [19], Ardema [20], Goering et.al. [21] and Ross et.al. [22]. The survey paper by Kadalbajoo and Reddy [23], gives an erudite outline of the singular perturbation problems. Few other notable methods for solving singular perturbation problems are finite difference methods [24-26], finite element methods [27], boundary value technique [28-30], initial value techniques [31-33], spline techniques [34-35], and so on.

In order to know the behavior of the solution of the singular perturbation problem in the boundary layer region, it is always suggestive to solve the problem in outer and boundary layer regions separately. In this paper, we present a numerical patching method for solving a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. By constructing a modified problem with a boundary layer correction, the boundary layer is dealt with separately and series method used. The condition at infinity will be applied to the corresponding Padé approximates of the obtained series solution. Several problems are solved to demonstrate the applicability and efficiency of the proposed method. It is observed that the present method approximates the exact solution very well.

2. A numerical patching method

For convenience we call our method the 'a numerical patching method'. We consider a class of linear singular perturbed two point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x)\ ; \quad x \in [0,1] \tag{1}$$

with the conditions $y(0) = \alpha$ and $y(1) = \beta$ (2)

where ε is a small positive parameter ($0 < \varepsilon << 1$), and α, β are known constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $a(x) \geq M > 0$, where M is some positive constant. This assumption merely implies that (1)-(2) has a solution which displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

We obtain the reduced problem by setting ε=0 in equation (1) and solve it for the solution with the appropriate boundary condition. Let $U(x)$ be the solution of the reduced problem of (1)-(2), i.e.;

$$a(x)U'(x) + b(x)U(x) = f(x) \quad \text{with} \tag{3}$$

$$U(1) = \beta. \tag{4}$$

It is well known from the singular perturbation theory that over most of the interval [0,1] the solution of (1)-(2) behaves like the solution of (3)-(4) but to satisfy the other boundary condition there is a small region in which the solution of (1)-(2) must deviate greatly from that of (3)-(4). This region is usually referred as boundary layer region. We choose so called stretching transformation $t = x/\varepsilon$.

This transforms the equation (1) into

$$\frac{d^2y}{dt^2} + a(t\varepsilon)\frac{dy}{dt} + \varepsilon\, b(t\varepsilon)y = \varepsilon\, f(t\varepsilon) \tag{5}$$

Upon setting $\varepsilon = 0$ in (5), we have

$$\frac{d^2y}{dt^2} + a(0)\frac{dy}{dt} = 0 \tag{6}$$

If we require the solution to equation (5) to compensate for the fact that the solution of the reduce problem (3)-(4) does not satisfy the boundary condition at $x = 0$, and further that this solution goes to zero as $t \to +\infty$, then we obtain the boundary layer correction problem

$$V''(t) + a(0)\ V'(t) = 0\ ;\quad t > 0 \tag{7}$$

with $V(0) = \alpha - U(0)$, $\lim\limits_{t\to\infty} V(t) = 0$ (8)

Then, from standard singular perturbation theory it follows that the solution of (1) and (2) admits the representation in terms of the solutions of the reduced and boundary layer correction problems. Thus we can write the solution of (1)-(2) as an asymptotic expansion:

$$y(x) = U(x) + V\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \tag{9}$$

as $\varepsilon \to 0$ uniformly in $[0,1]$, with U the solution of (3)-(4) and V the solution of (7)- (8). The usual numerical methods can not be directly applied to (7)-(8) with out some modification. As will be seen, out boundary layer correction problem is one of such modification. The idea of our method is to construct U and V in (9) such that the solution $y(x) = U(x) + V\left(\frac{x}{\varepsilon}\right)$ can be used to approximate the solution $y(x)$ of (1)-(2). There is no perturbation parameter ε in (3)-(4), so it is easy to get the numerical solution by Runge-Kutta method. In fact any other standard analytical or numerical method can be used to solve the reduced problem (3)-(4). Although ε does not appear explicitly in the boundary layer correction problem (7)-(8), the semi infinite domain causes some difficulty. So we use the following technique:
Let $V'(0) = \gamma$ and we will determine the value of γ by using the condition $\lim\limits_{t\to\infty} V(t) = 0$.

Thus (7) and (8) could be substituted by the initial problem,

$$V''(t) + a(0)\ V'(t) = 0\ ;\quad t > 0 \tag{10}$$

with $V(0) = \alpha - U(0)$, $V'(0) = \gamma$ (11)

The equation (10) is a constant coefficient linear differential equation and its solution is given by

$$V(t) = \frac{-\gamma}{a(0)}e^{-a(0)t} + (\alpha - U(0)) + \frac{\gamma}{a(0)}$$

By using the condition $\lim\limits_{t\to\infty} V(t) = 0$, we get $\gamma = -a(0)(\alpha - U(0))$.

Padé Approximation

When the function V(t) is such that it remains zero as t tends to infinity, the polynomial approximations give very poor results. In comparison, rational approximations (Padé approximations) give much better result [36].

Now we will explain that the value of γ can be also obtained by Padé approximate and series method. Rewrite V(t) to a series form, $V(t)=\sum_{n=0}^{\infty} a_n t^n$, substitute it into equations (7)-(8), we obtain

$$\mathrm{V(t)} = \alpha - \mathrm{U(0)} + \gamma \mathrm{t} - \frac{1}{2}\mathrm{a(0)}\gamma \mathrm{t}^2 + \frac{1}{6}(\mathrm{a(0)})^2 \gamma \mathrm{t}^3 - ..... \tag{12}$$

Because we could not use the condition $\mathrm{V}(\infty)=0$ directly, we use Pade' approximate [L/M]

$$[L/M] = \frac{P_L(t)}{Q_M(t)}, \tag{13}$$

where $\mathrm{P_L(t), Q_M(t)}$ are polynomials of degrees at most L and M respectively. Besides, we may consider $\mathrm{Q_M(0)}=1$, and $\mathrm{P_L(t)}$ and $\mathrm{Q_M(t)}$ have no common factors. In the following we will determine the Pade' approximates [2/2] of (12). The Pade' approximates [3/3] and [4/4] can be determined in a parallel manner.

$$V(t) = \frac{A_0 + A_1 t + A_2 t^2}{1 + B_1 t + B_2 t^2}.$$

To determine the Pade' approximates [2/2] to V(t) of degree 4, it requires choosing $\mathrm{A_0, A_1, A_2, B_1, B_2}$ so that the coefficients of $\mathrm{t^k}$ for $\mathrm{k} = 0,1,2,3,4$ are zero in the expression

$$\mathrm{V(t)}\left(1 + \mathrm{B_1 t} + \mathrm{B_2 t^2}\right) = \mathrm{A_0} + \mathrm{A_1 t} + \mathrm{A_2 t^2}. \tag{14}$$

Expanding (14) and equating the coefficients of $\mathrm{t^k}$ for $\mathrm{k} = 0,1,2,3,4$ to zero yields

$$B_1 = a(0),$$
$$B_2 = \frac{1}{3}(a(0))^2,$$
$$A_0 = \alpha - U(0),$$
$$A_1 = (\alpha - U(0))a(0) + \gamma,$$
$$A_2 = \frac{1}{3}(a(0))^2(\alpha - U(0)) + \frac{1}{2}a(0)\gamma.$$

So we get [2/2] to V(t) as $V(t) = \alpha - U(0) + \dfrac{\gamma t + \frac{1}{2}a(0)\gamma t^2}{1 + a(0)t + \frac{1}{2}(a(0))^2 t^2}$.

Using $\mathrm{V}(\infty)=0$, we get

$$\gamma = -a(0)(\alpha - U(0)). \tag{15}$$

It is the same result as shown for the linear case.

2.1. Linear examples

Example 1: Consider the following homogeneous singular perturbation problem

$\varepsilon y''(x) + y'(x) - y(x) = 0$, $x \in [0,1]$

with the boundary conditions $y(0) = 1$ and $y(1) = 1$

The exact solution is given by $y(x) = \dfrac{\left[\left(e^{m_2} - 1\right)e^{m_1} + \left(1 - e^{m_1}\right)e^{m_2}\right]}{\left[e^{m_2} - e^{m_1}\right]}$.

The reduced problem is $U'(x) - U(x) = 0$ with $U(1) = 1$

and its solution is $U(x) = e^{x-1}$.

The boundary layer correction problems is

$V''(t) + V'(t) = 0$, with $V(0) = \alpha - U(0)$, $V'(0) = \gamma$.

Using the condition $\lim_{t\to\infty} V(t) = 0$ we obtain $\gamma = e^{-1} - 1$.

Thus we obtain $V(t) = (1 - e^{-1}) + \left(e^{-1} - 1\right)(1 - e^{-t})$, where $t = \dfrac{x}{\varepsilon}$.

The required solution is $y(x) = e^{x-1} + (1 - e^{-1})e^{-\frac{x}{\varepsilon}}$.

The numerical results are given in Tables 1(a), 1(b) for $\varepsilon=10^{-5}$ and 10^{-7} respectively.

Table 1. (a) $\varepsilon = 10^{-5}, h = 10^{-3}$; (b) $\varepsilon = 10^{-7}, h = 10^{-3}$

(a) X	y(x)	Exact Solution	(b) X	y(x)	Exact Solution
0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000
0.0020000	0.3686159	0.3681204	0.0020000	0.3686159	0.4097199
0.0040000	0.3693539	0.3688583	0.0040000	0.3693539	0.4104533
0.0060000	0.3700933	0.3695978	0.0060000	0.3700933	0.4111879
0.0080000	0.3708343	0.3703387	0.0080000	0.3708343	0.4119238
0.0100000	0.3715767	0.3710811	0.0100000	0.3715767	0.4126610
0.1000000	0.4065697	0.4060767	0.1000000	0.4065697	0.4472388
0.2000000	0.4493290	0.4488447	0.2000000	0.4493290	0.4890670
0.3000000	0.4965853	0.4961170	0.3000000	0.4965853	0.5348073
0.4000000	0.5488116	0.5483679	0.4000000	0.5488116	0.5848255
0.5000000	0.6065307	0.6061220	0.5000000	0.6065307	0.6395217
0.6000000	0.6703200	0.6699587	0.6000000	0.6703200	0.6993333
0.7000000	0.7408183	0.7405187	0.7000000	0.7408183	0.7647389
0.8000000	0.8187308	0.8185101	0.8000000	0.8187308	0.8362616
0.9000000	0.9048374	0.9047155	0.9000000	0.9048374	0.9144734
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Example 2: Consider the non homogeneous singular perturbation problem from fluid dynamics of small viscosity [37]

$\varepsilon y''(x) + y'(x) = 1 + 2x$, $x \in [0,1]$ with the boundary conditions $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + \dfrac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-1/\varepsilon})}$.

The reduced problem is $U'(x) = 1 + 2x$, $U(1) = 1$.

Its solution is $U(x) = x^2 + x - 1$.

The boundary layer correction problem is $V''(t) + V'(t) = 0$, with $V(0) = 1$ and $V'(0) = \gamma$.

Using the condition $\lim_{t \to \infty} V(t) = 0$ we obtain $\gamma = -1$.

Hence $V(t) = e^{-t}$.

The required solution is $y(x) = x^2 + x - 1 + e^{-\frac{x}{\varepsilon}}$.

The numerical results are given in Tables 2(a), 2(b) for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Table 2. (a) $\varepsilon = 10^{-5}, h = 10^{-3}$; (b) $\varepsilon = 10^{-7}, h = 10^{-3}$

(a) X	y(x)	Exact Solution	(b) X	y(x)	Exact Solution
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.0020000	-0.9979960	-0.9979760	0.0020000	-0.9979960	-0.9979958
0.0040000	-0.9959840	-0.9959641	0.0040000	-0.9959840	-0.9959838
0.0060000	-0.9939640	-0.9939441	0.0060000	-0.9939640	-0.9939638
0.0080000	-0.9919360	0.9919161	0.0080000	-0.9919360	-0.9919358
0.0100000	-0.9899000	-0.9898801	0.0100000	-0.9899000	-0.9898998
0.1000000	-0.8900000	-0.8899820	0.1000000	-0.8900000	-0.8899999
0.2000000	-0.7600000	-0.7599840	0.2000000	-0.7600000	-0.7599999
0.3000000	-0.6100000	-0.6099859	0.3000000	-0.6100000	-0.6099999
0.4000000	-0.4400000	-0.4399880	0.4000000	-0.4400000	-0.4399999
0.5000000	-0.2500000	-0.2499900	0.5000000	-0.2500000	-0.2499999
0.6000000	-0.0399999	-0.0399919	0.6000000	-0.0399999	-0.0399999
0.7000000	0.1900001	0.1900061	0.7000000	0.1900001	0.1900001
0.8000000	0.4400000	0.4400041	0.8000000	0.4400000	0.4400000
0.9000000	0.7100001	0.7100022	0.9000000	0.7100001	0.7100001
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	0.9999999

Example 3: Consider the variable coefficient singular perturbation problem [3]

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0 , \quad x \in [0,1]$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x)=\frac{1}{2-x}-\frac{1}{2}e^{\frac{-\left(x-\frac{x^2}{4}\right)}{\varepsilon}}$.

The reduced problem is $\left(1-\frac{x}{2}\right)U'(x)-\frac{1}{2}U(x)=0$, $U(1)=1$ and

its solution is $U(x)=\frac{1}{2-x}$.

The boundary layer correction problem is

$V''(t)+V'(t)=0$, *with* $V(0)=\frac{-1}{2}$ and $V'(0)=\gamma$.

Using the condition $\lim_{t\to\infty} V(t)=0$ we obtain $\gamma=1/2$.

$\therefore V(t)=-\frac{1}{2}e^{-t}$.

The required solution is $y(x)=\frac{1}{2-x}-\frac{1}{2}e^{-\frac{x}{\varepsilon}}$.

The numerical results are given in Tables 3(a), 3(b) for $\varepsilon=10^{-5}$ and 10^{-7} respectively.

Table 3. (a) $\varepsilon=10^{-5}, h=10^{-3}$; (b) $\varepsilon=10^{-7}, h=10^{-3}$

(a)			(b)		
X	y(x)	Exact Solution	X	y(x)	Exact Solution
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.0020000	0.5005005	0.5005005	0.0020000	0.5005005	0.5005005
0.0040000	0.5010020	0.5010020	0.0040000	0.5010020	0.5010020
0.0060000	0.5015045	0.5015045	0.0060000	0.5015045	0.5015045
0.0080000	0.5020080	0.5020080	0.0080000	0.5020080	0.5020080
0.0100000	0.5025126	0.5025126	0.0100000	0.5025126	0.5025126
0.1000000	0.5263158	0.5263158	0.1000000	0.5263158	0.5263158
0.2000000	0.5555556	0.5555556	0.2000000	0.5555556	0.5555556
0.3000000	0.5882353	0.5882353	0.3000000	0.5882353	0.5882353
0.4000000	0.6250000	0.6250000	0.4000000	0.6250000	0.6250000
0.5000000	0.6666667	0.6666667	0.5000000	0.6666667	0.6666667
0.6000000	0.7142857	0.7142857	0.6000000	0.7142857	0.7142857
0.7000000	0.7692308	0.7692308	0.7000000	0.7692308	0.7692308
0.8000000	0.8333333	0.8333333	0.8000000	0.8333333	0.8333333
0.9000000	0.9090909	0.9090909	0.9000000	0.9090909	0.9090909
1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

2.2. Nonlinear singular perturbation problems

We consider a class of nonlinear singular perturbed two point boundary value problem of the form

$$\varepsilon y''(x) + f(x,y)y'(x) + g(x,y) = 0\ ;\quad 0 < x < 1 \tag{16}$$

with the boundary conditions

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta \tag{17}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$), and α, β are known constants.

We assume that $f(x,y)$ and $g(x,y)$ are sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $f(x,y) \geq M > 0$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$.

The reduced problem is

$$f(x,U)U'(x) + g(x,U) = 0 \quad \text{with} \quad U(1) = \beta \tag{18}$$

Using the stretching transformation $t = x/\varepsilon$ equation (16) reduces to

$$\frac{d^2y}{dt^2} + f(t\varepsilon, y)\frac{dy}{dt} + \varepsilon g(t\varepsilon, y) = 0. \tag{19}$$

Setting $\varepsilon = 0$ in (19) we have

$$\frac{d^2y}{dt^2} + f(0,y)\frac{dy}{dt} = 0. \tag{20}$$

If we require the solution to (20) to compensate for the fact that the solution of the reduce problem (18) does not satisfy the boundary condition at $x = 0$, and further that this solution goes to zero as $t \to +\infty$, then we obtain the boundary layer correction problem

$$V''(t) + f(0, U(0) + V)\ V'(t) = 0\ ;\quad t > 0 \tag{21}$$

$$\text{with} \quad V(0) = \alpha - U(0) \quad , \quad \lim_{t\to\infty} V(t) = 0 \tag{22}$$

Then, from standard singular perturbation theory it follows that the solution of (16) and (17) admits the representation in terms of the solutions of the reduced and boundary layer correction problems. Thus we can write the solution of (16)-(17) as an asymptotic expansion:

$$y(x) = U(x) + V\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Although there is no ε in (18) and (21), it is difficult to apply the condition $\lim_{t\to\infty} V(t) = 0$, so we use the following technique in shooting method. Let $V'(0) = \gamma$ and we will determine the value of γ by the condition $\lim_{t\to\infty} V(t) = 0$.

Thus equations (21) and (22) could be substituted by the initial problem,

$$V''(t) + f(0, U(0) + V)\ V'(t) = 0\ ;\quad t > 0 \tag{23}$$

$$\text{with} \quad V(0) = \alpha - U(0) \quad , \quad V'(0) = \gamma \tag{24}$$

For nonlinear problems, we can not solve (23)-(24). Thus we use approximate method as (12)-(15). Form the numerical results it has been observed that the method is effective.

2.3. Nonlinear examples

Example 4: Consider the nonlinear singular perturbation problem [2]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0 \ , \ x \in [0,1]$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$.

The exact solution is given by $y(x) = \log_e\left(\frac{2}{1+x}\right) - (\log_e 2)e^{-2x/\varepsilon}$.

The reduced problem is $2U' + e^U = 0$ with $U(1) = 0$.

And its solution is $U(x) = -\ln\left(\frac{x}{2} + \frac{1}{2}\right)$.

The boundary layer correction problems is

$V''(t) + 2V'(t) = 0$, *with* $V(0) = \ln(1/2)$ and $V'(0) = \gamma$.

Using the condition $\lim_{t\to\infty} V(t) = 0$ we obtain $\gamma = -\frac{1}{2}\log\left(\frac{1}{2}\right)$.

$\therefore V(t) = \ln(1/2)e^{-2t}$.

The required solution is $y(x) = -\ln\left(\frac{x}{2} + \frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)e^{-2x/\varepsilon}$.

The numerical results are given in Tables 4(a), 4(b) for $\varepsilon=10^{-3}$ and 10^{-4} respectively.

Table 4. (a) $\varepsilon = 10^{-5}, h = 10^{-3}$; (b) $\varepsilon = 10^{-7}, h = 10^{-3}$

(a) X	Y	Exact Solution	(b) X	Y	Exact Solution
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.0020000	0.6911492	0.6911492	0.0020000	0.6911492	0.6911492
0.0040000	0.6891552	0.6891552	0.0040000	0.6891552	0.6891552
0.0060000	0.6871651	0.6871651	0.0060000	0.6871651	0.6871651
0.0080000	0.6851790	0.6851790	0.0080000	0.6851790	0.6851790
0.0100000	0.6831968	0.6831968	0.0100000	0.6831968	0.6831968
0.1000000	0.5978370	0.5978370	0.1000000	0.5978370	0.5978370
0.2000000	0.5108256	0.5108256	0.2000000	0.5108256	0.5108256
0.3000000	0.4307829	0.4307829	0.3000000	0.4307829	0.4307829
0.4000000	0.3566749	0.3566749	0.4000000	0.3566749	0.3566749
0.5000000	0.2876821	0.2876821	0.5000000	0.2876821	0.2876821
0.6000000	0.2231435	0.2231435	0.6000000	0.2231435	0.2231435
0.7000000	0.1625189	0.1625189	0.7000000	0.1625189	0.1625189
0.8000000	0.1053605	0.1053605	0.8000000	0.1053605	0.1053605
0.9000000	0.0512933	0.0512933	0.9000000	0.0512933	0.0512933
1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	0.0000000

Example 5: Consider the nonlinear singular perturbation problem [3]

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0 \ , \quad x \in [0,1]$$

with the boundary conditions $y(0) = -1$ and $y(1) = 3.9995$.

The exact solution is given by $y(x) = x + c_1 \tanh\left(\dfrac{c_1\left(\dfrac{x}{\varepsilon} + c_2\right)}{2}\right)$

where $c_1 = 2.9995$ *and* $c_2 = \left(\dfrac{1}{c_1}\right)\log_e\left(\dfrac{c_1 - 1}{c_1 + 1}\right)$.

The reduced problem is $\mathrm{U}\mathrm{U}' - \mathrm{U} = 0$, $\mathrm{U}(1) = 3.9995$

and its solution is $\mathrm{U}(\mathrm{x}) = \mathrm{x} + 2.9995$.

The boundary layer correction problems is

$\mathrm{V}''(\mathrm{t}) + (\mathrm{U}(0) + \mathrm{V})\mathrm{V}'(\mathrm{t}) = 0$, with $\mathrm{V}(0) = -3.9995$ and $\mathrm{V}'(0) = \gamma$.

We have to obtain γ using the condition $\lim_{t\to\infty} V(t) = 0$.

Taking the series $\mathrm{V}(\mathrm{t}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{a}_{\mathrm{n}}\, \mathrm{t}^{\mathrm{n}}$ and using it in boundary layer correction problem, we get

$$a_0 = -3.9995 \ , \quad a_1 = \gamma \ , \quad a_2 = \frac{\gamma}{2} \ , \quad a_3 = \frac{1}{6}(\gamma - \gamma^2) \ , \quad a_4 = \frac{1}{24}(\gamma - 4\gamma^2)$$

$$a_5 = \frac{1}{20}\left(\frac{2}{3}\gamma^3 + \frac{1}{6}\gamma - \frac{11}{6}\gamma^2\right) \ , \quad a_6 = \frac{1}{720}\left(34\gamma^3 + \gamma - 26\gamma^2\right).$$

$$\therefore V(t) = -3.9995 + \gamma t + \frac{\gamma}{2}t^2 + \frac{1}{6}(\gamma - \gamma^2)t^3 + \frac{1}{24}(\gamma - 4\gamma^2)t^4 + \frac{1}{20}\left(\frac{2}{3}\gamma^3 + \frac{1}{6}\gamma - \frac{11}{6}\gamma^2\right)t^5 +$$

$$\frac{1}{720}\left(34\gamma^3 + \gamma - 26\gamma^2\right)t^6 + \ldots\ldots\ldots\ldots$$

Using the Pade' approximate [3/3]

$$V(t) = \frac{A_0 + A_1 t + A_2 t^2 + A_3 t^3}{1 + B_1 t + B_2 t^2 + B_3 t^3}, \text{ we get}$$

$$A_0 = -3.9995$$

$$A_1 = -3.9995\ B_1 + \gamma$$

$$A_2 = -3.9995\ B_2 + \gamma\, B_1 + \frac{\gamma}{2}$$

$$A_3 = -3.9995\ B_3 + \gamma\, B_2 + \frac{\gamma}{2}B_1 + \frac{1}{6}(\gamma - \gamma^2)$$

$$\frac{1}{6}(\gamma - \gamma^2)B_1 + \frac{\gamma}{2}B_2 + \gamma\, B_3 + \frac{1}{24}(\gamma - 4\gamma^2) = 0$$

$$\frac{1}{24}(\gamma - 4\gamma^2)B_1 + \frac{1}{6}(\gamma - \gamma^2)B_2 + \frac{\gamma}{2}B_3 + \frac{1}{20}\left(\frac{2}{3}\gamma^3 + \frac{1}{6}\gamma - \frac{11}{6}\gamma^2\right) = 0$$

$$\frac{1}{20}\left(\frac{2}{3}\gamma^3 + \frac{1}{6}\gamma - \frac{11}{6}\gamma^2\right)B_1 + \frac{1}{24}(\gamma - 4\gamma^2)B_2 + \frac{1}{6}(\gamma - \gamma^2)B_3 + \frac{1}{720}\left(34\gamma^3 + \gamma - 26\gamma^2\right) = 0.$$

On solving, we get

$$A_0 = -3.9995$$

$$A_1 = 1.9995 + \gamma$$

$$A_2 = -0.39995(1 + 2\gamma)$$

$$A_3 = 0.03332916667(1 + 2\gamma) + 0.1\gamma(1 + 2\gamma) - 0.25\gamma + 0.166667\gamma(1 - \gamma)$$

$$B_1 = -\frac{1}{2}$$

$$B_2 = \frac{1}{10}(1 + 2\gamma)$$

$$B_3 = \frac{-1}{120}(1 + 2\gamma)$$

$$V(t) = \frac{-3.9995 + (1.9995 + \gamma)t + [-0.39995(1 + 2\gamma)]t^2 + [0.03332916667(1 + 2\gamma) + 0.1\gamma(1 + 2\gamma) - 0.25\gamma + 0.166667\gamma(1 - \gamma)]t^3}{1 + (-0.5)t + [0.1(1 + 2\gamma)]t^2 + [-0.008333333(1 + 2\gamma)]t^3}$$

On applying the condition $\lim_{t \to \infty} V(t) = 0$ we get $\gamma = -1.999791999$.

The required solution $y(x) = U(x) + V(t)$.

The numerical results are given in Tables 5 for $\varepsilon = 10^{-5}$ and 10^{-7} respectively.

Table 5. (a) $\varepsilon = 10^{-5}$, $h = 10^{-3}$; (b) $\varepsilon = 10^{-7}$, $h = 10^{-3}$

(a) X	Y	Exact Solution	(b) X	Y	Exact Solution
0.0000000	-1.0000000	-1.0000000	0.0000000	-1.0000000	-1.0000000
0.0020000	3.2561400	3.0015001	0.0020000	3.0031865	3.0015001
0.0040000	3.1264725	3.0035000	0.0040000	3.0039856	3.0035000
0.0060000	3.0863974	3.0055001	0.0060000	3.0055857	3.0055001
0.0080000	3.0676823	3.0074999	0.0080000	3.0073855	3.0074999
0.0100000	3.0573547	3.0095000	0.0100000	3.0092657	3.0095000
0.1000000	3.1035900	3.0994999	0.1000000	3.0988336	3.0994999
0.2000000	3.2011864	3.1995001	0.2000000	3.1988096	3.1995001
0.3000000	3.3003857	3.2995000	0.3000000	3.2988017	3.2995000
0.4000000	3.3999858	3.3995001	0.4000000	3.3987978	3.3995001
0.5000000	3.4997456	3.4995000	0.5000000	3.4987953	3.4995000
0.6000000	3.5995858	3.5995002	0.6000000	3.5987937	3.5995002
0.7000000	3.6994715	3.6995001	0.7000000	3.6987927	3.6995001
0.8000000	3.7993855	3.7995000	0.8000000	3.7987916	3.7995000
0.9000000	3.8993189	3.8995001	0.9000000	3.8987911	3.8995001
1.0000000	3.9992657	3.9995000	1.0000000	3.9987905	3.9995000

3. Right-end boundary layer problems

We consider a class of non linear singular perturbed two point boundary value problem of the form

$$\varepsilon y''(x) + f(x,y)y'(x) + g(x,y) = 0\ ;\quad 0 < x < 1 \tag{25}$$

with the conditions.

$$y(0) = \alpha \text{ and } y(1) = \beta \tag{26}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$), and α, β are known constants.

We assume that $f(x,y)$ and $g(x,y)$ are sufficiently continuously differentiable functions in [0,1]. Furthermore, we assume that $f(x,y) \le M < 0$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

The reduced problem is

$$f(x,U)U'(x) + g(x,U) = 0 \quad \text{with} \quad U(0) = \alpha. \tag{27}$$

Using the stretching transformation $t = \dfrac{1-x}{\varepsilon}$ equation (25) to

$$\frac{d^2y}{dt^2} + f(1-t\varepsilon, y)\frac{dy}{dt} + \varepsilon g(1-t\varepsilon, y) = 0.$$

As $\varepsilon \to 0$ we have

$$\frac{d^2y}{dt^2} + f(1,y)\frac{dy}{dt} = 0 \tag{28}$$

If we require the solution to (28) to compensate for the fact that the solution of the reduce problem (27) does not satisfy the boundary condition at $x = 1$, and further that this solution goes to zero as $t \to +\infty$, then we obtain the boundary layer correction problem

$$V''(t) + f(1, U(1) + V)\ V'(t) = 0\ ;\quad t > 0 \tag{29}$$

$$\text{with} \quad V(1) = \beta - U(1)\ ,\quad \lim_{t\to\infty} V(t) = 0 \tag{30}$$

Then, from standard singular perturbation theory it follows that the solution of (25) and (26) admits the representation in terms of the solutions of the reduced and boundary layer correction problems. Thus we can write the solution as an asymptotic expansion:

$$y(x) = U(x) + V\left(\frac{1-x}{\varepsilon}\right) + O(\varepsilon) \tag{31}$$

as $\varepsilon \to 0$ uniformly in [0,1], with U the solution of (27) and V the solution of (29)-(30).

The idea of our method is to construct U and V such that the solution

$$y(x) = U(x) + V\left(\frac{1-x}{\varepsilon}\right) \tag{32}$$

can be used to approximate the solution $y(x)$ of (25)-(26). There is no perturbation parameter ε in (27), so it is easy to get either analytical solution or numerical solution. Although there is no ε in (29)-(30), it is difficult to apply the condition $\lim_{t\to\infty} V(t) = 0$, so we use the following technique in shooting method. Let $V'(1) = \gamma$ and we will determine the value of γ by the condition $\lim_{t\to\infty} V(t) = 0$.

Thus equations (29)-(30) could be substituted by the initial problem,

$$V''(t) + f(1, U(1) + V)\, V'(t) = 0\ ; \quad t > 0 \tag{33}$$

with $V(1) = \beta - U(1)$, $V'(1) = \gamma$. (34)

Now we discuss the solution of (33)-(34). We use the series method. Let $V(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$, and apply it into (33)-(34). We could get the value of $a_n\,(n = 0,1,2,.....)$, and of course it is a expression corresponding to γ. Thus $V(t)$ could be a series about γ and t. Using the condition $\lim_{t\to\infty} V(t) = 0$ and rational approximate, we get the series solution of (33)-(34).

Through $y(x) = U(x) + V(t)$, we get the approximate solution of (25)-(26).

3.1. Examples with right -end boundary layer

Example 6: Consider the homogeneous singular perturbation problem [32]

$$\varepsilon y''(x) - y'(x) = 0\ ,\ x \in [0,1]$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$.

The exact solution is given by $y(x) = \dfrac{\left(e^{(x-1)/\varepsilon} - 1\right)}{\left(e^{-1/\varepsilon} - 1\right)}$.

The reduced problem is $U'(x) = 0$ with $U(0) = 1$.
Its solution is $U(x) = 1$.
The boundary layer correction problem is

$V''(t) + V'(t) = 0$, with $V(0) = -1$, $V'(0) = \gamma$.

Using the condition $\lim_{t\to\infty} V(t) = 0$ we obtain $\gamma = 1$.

$\therefore V(t) = -e^{-t}$.

We have $y(x) = U(x) + V(t)$.

Therefore the required solution is $y(x) = 1 - e^{-\frac{(1-x)}{\varepsilon}}$.

The numerical results are given in Tables 6(a), 6(b) for $\varepsilon=10^{-5}$ and 10^{-7} respectively.

Example 7: Consider the homogeneous singular perturbation problem [32]

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0\ ,\ x \in [0,1]$$

with the boundary conditions $y(0) = 1 + \exp(-(1+\varepsilon)\varepsilon^{-1})$ and $y(1) = 1 + e^{-1}$.

The exact solution is given by $y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$.

The reduced problem is $-U'(x) - U(x) = 0$ with $U(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$.

On solving we get $U(x) = [1 + \exp(-(1+\varepsilon)/\varepsilon)]e^{-x}$.

The boundary layer correction problem is, $V''(t) + V'(t) = 0$

with $V(0) = 1 - \exp(-(1+2\varepsilon)/\varepsilon)$ and $V'(0) = \gamma$.

Using the condition $\lim_{t\to\infty} V(t) = 0$ we obtain $\gamma = e^{-(1+2\varepsilon)/\varepsilon} - 1$.

Hence $V(t) = [1 - \exp(-(1+2\varepsilon)/\varepsilon)]e^{-t}$.

The required solution is

$$y(x) = [1 + \exp(-(1+\varepsilon)/\varepsilon)]e^{-x} + [1 - \exp(-(1+2\varepsilon)/\varepsilon)]e^{-\frac{(1-x)}{\varepsilon}}.$$

The numerical results are given in Tables 7(a), 7(b) for $\varepsilon=10^{-5}$ and 10^{-7} respectively.

Table 6. (a) $\varepsilon = 10^{-5}, h = 10^{-3}$; (b) $\varepsilon = 10^{-7}, h = 10^{-3}$

(a) X	Y	Exact Solution	(b) X	Y	Exact Solution
0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000
0.1000000	1.0000000	1.0000000	0.1000000	1.0000000	1.0000000
0.2000000	1.0000000	1.0000000	0.2000000	1.0000000	1.0000000
0.3000000	1.0000000	1.0000000	0.3000000	1.0000000	1.0000000
0.4000000	1.0000000	1.0000000	0.4000000	1.0000000	1.0000000
0.5000000	1.0000000	1.0000000	0.5000000	1.0000000	1.0000000
0.6000000	1.0000000	1.0000000	0.6000000	1.0000000	1.0000000
0.7000000	1.0000000	1.0000000	0.7000000	1.0000000	1.0000000
0.8000000	1.0000000	1.0000000	0.8000000	1.0000000	1.0000000
0.9000000	1.0000000	1.0000000	0.9000000	1.0000000	1.0000000
0.9200000	1.0000000	1.0000000	0.9200000	1.0000000	1.0000000
0.9300001	1.0000000	1.0000000	0.9300001	1.0000000	1.0000000
0.9400001	1.0000000	1.0000000	0.9400001	1.0000000	1.0000000
0.9500000	1.0000000	1.0000000	0.9500000	1.0000000	1.0000000
0.9600000	1.0000000	1.0000000	0.9600000	1.0000000	1.0000000
0.9700000	1.0000000	1.0000000	0.9700000	1.0000000	1.0000000
0.9800000	1.0000000	1.0000000	0.9800000	1.0000000	1.0000000
0.9900001	1.0000000	1.0000000	0.9900001	1.0000000	1.0000000
1.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000

Table 7. (a) $\varepsilon = 10^{-5}, h = 10^{-3}$; (b) $\varepsilon = 10^{-7}, h = 10^{-3}$

(a) X	Y	Exact Solution	(b) X	Y	Exact Solution
0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000
0.1000000	0.9048374	0.9048374	0.1000000	0.9048374	0.9048374
0.2000000	0.8187308	0.8187308	0.2000000	0.8187308	0.8187308
0.3000000	0.7408182	0.7408182	0.3000000	0.7408182	0.7408182
0.4000000	0.6703200	0.6703200	0.4000000	0.6703200	0.6703200
0.5000000	0.6065307	0.6065307	0.5000000	0.6065307	0.6065307
0.6000000	0.5488116	0.5488116	0.6000000	0.5488116	0.5488116
0.7000000	0.4965853	0.4965853	0.7000000	0.4965853	0.4965853
0.8000000	0.4493290	0.4493290	0.8000000	0.4493290	0.4493290
0.9000000	0.4065697	0.4065697	0.9000000	0.4065697	0.4065697
0.9200000	0.3985190	0.3985190	0.9200000	0.3985190	0.3985190
0.9300001	0.3945537	0.3945537	0.9300001	0.3945537	0.3945537
0.9400001	0.3906278	0.3906278	0.9400001	0.3906278	0.3906278
0.9500000	0.3867410	0.3867410	0.9500000	0.3867410	0.3867410
0.9600000	0.3828929	0.3828929	0.9600000	0.3828929	0.3828929
0.9700000	0.3790830	0.3790830	0.9700000	0.3790830	0.3790830
0.9800000	0.3753111	0.3753111	0.9800000	0.3753111	0.3753111
0.9900000	0.3715767	0.3715767	0.9900000	0.3715767	0.3715767
1.0000000	1.3678794	1.3678794	1.0000000	1.3678794	1.3678794

4. Conclusions

We have described the a numerical patching method for solving a class of singularly perturbed two point boundary value problems with a boundary layer at one end point. It provides an alternate and supplementary method to the conventional ways of solving certain class of singular perturbation problems. It is a practical method, can be implemented on a computer with a modest amount of problem preparation. We have implemented the present method on three linear examples, two non-linear examples with left-end boundary layer and two examples with right-end boundary layer by taking different values of ε. The approximate solution is compared with exact solution. It can be observed from the results that the present method agrees with exact solution very well, which shows the efficiency of the method.

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