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Computational Method for Solving Singularly Perturbed Delay Differential Equations with Negative Shift

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Abstract: In this paper, a computational method is presented for solving singularly perturbed delay differential equations with negative shift whose solution has boundary layer. First, the second order singularly perturbed delay differential equation is replaced by an asymptotically equivalent first order delay differential equation. Then, Trapezoidal integration formula and linear interpolation are employed to get three term recurrence relation which is solved easily by Discrete Invariant Imbedding Algorithm. The method is demonstrated by implementing several model examples by taking various values for the delay parameter and the perturbation parameter.

Keywords: Singular perturbations; delay differential equations; delay parameter; boundary layer; perturbation parameter; trapezoidal rule.

1. Introduction

Consider the singularly perturbed differential difference equation

$$\varepsilon y'' + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 < x < 1 \tag{1}$$

under the interval and boundary conditions

$$y(x) = \phi(x), \quad -\delta \le x \le 0, \quad y(1) = \gamma . \tag{2}$$

where $a(x), b(x) \le -\theta < 0$ and $f(x)$ are sufficiently smooth functions, $0 < \varepsilon << 1$ and $\delta = o(\varepsilon)$ such that $(\varepsilon - \delta a(x)) > 0$ for all $x \in [0,1]$. Furthermore, γ and θ are positive constants. Such problems are found throughout the literature on epidemics and population where this small shift plays an important role in modeling of various real life phenomena (Kuang [8]). For example, in the mathematical model for the determination of the expected first-exist time in the generation of action potential in nerve cells by random synaptic inputs in dendrites (Lange and Miura [9]) and description of the human pupil-light reflex (Longtin and Milton [11]). The shifts are due to the jumps in the potential membrane which are very small. These biological problems motivate the study of the boundary value problems for singularly perturbed differential difference equations with delay as well as advance, which was initiated by Lange and Miura [9, 10], where they introduced the new terminology 'negative shift' for 'delay' and 'positive shift' for 'advance'. Hence in the recent times, many researchers have been trying to develop numerical methods for solving these problems. Kadalbajoo and et al [5] constructed and analyzed a fitted operator finite difference method to solve problems arising from singularly perturbed general differential difference equations. Patidar and Sharma [12] presented some

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uniformly convergent non standard finite difference methods for solving class of singularly perturbed differential difference equations where there is small delay in the convention term. Moreover, a wide verity of papers have been published in the recent years describing various methods for solving singularly perturbed delay differential equations, among these, we mention Phaneendra *et al* [13], Yadaw and Kadalbajoo [15], Kadalbajoo and Sharma [4]; and Rao and Chakravarthy [14]. Furthermore, many researchers have often observed that the shifts/delay parameters are very small and affect the solution very significantly. Lange and Miura [10] have shown that the effect of very small shifts (of order ε) on the solution and pointed out that they drastically affect the solution and therefore cannot be neglected. Further studies about the effect of the delay parameter, δ on the layer behavior of the solution has been carried out by Kadalbajoo and Sharma [6, 7].

There is a wide variety of asymptotic expansion methods available for solving singular perturbation problems. But there can be difficulties in applying these asymptotic expansion methods as finding of the appropriate asymptotic expansions in the inner and outer regions is not routine exercises rather requires skill, insight, and experimentations. Even the matching of the coefficients of the inner and outer solution expansions can be demanding process. Thus, one can raise the question that whether there may be other better ways that are easy and ready for computer implementation to attack these problems. In this paper, we have presented a numerical method that does not depend on the asymptotic expansion and matching of the coefficients for solving a class of singularly perturbed delay differential equations with negative shift. First, the second order singularly perturbed delay differential equation is replaced by an asymptotically equivalent first order delay differential equation. Then, Trapezoidal rule and linear interpolation are employed to get three term recurrence relation which is solved easily by discrete Invariant Imbedding algorithm. The method is demonstrated by implementing it on several model examples by taking various values for the delay and perturbation parameters.

2. Description of the method

2.1. Left-end boundary layer problems

Consider singularly perturbed boundary value problems of the form

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), 0 \le x \le 1 \tag{3}$$

with boundary conditions

$$y(0) = \alpha\,,\ -\delta \le x \le 0 \tag{4a}$$

and $y(1) = \beta$ (4b)

where $0 < \varepsilon << 1$, *b(x), f(x)* are bounded continuous functions in (0, 1) and α, β are finite constants. Further, we assume that $a(x) \ge M > 0$ throughout the interval [0, 1], where M is positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$.

By using Taylor series expansion in the neighborhood of the point *x,* we have

$$y\left(x - \sqrt{\varepsilon}\right) = y(x) - \sqrt{\varepsilon}\, y'(x) + \frac{\varepsilon}{2} y''(x) \tag{5}$$

and consequently, equation (3) is replaced by the following first order differential equation:

$$y'(x) = p(x)y(x - \sqrt{\varepsilon}) + q(x)y'(x-\delta) + r(x)y + s(x) \tag{6}$$

for $0 \le \delta < 1$, where

$$p(x) = \frac{-1}{\sqrt{\varepsilon}},\ q(x) = \frac{-a(x)}{2\sqrt{\varepsilon}},\ r(x) = \frac{2-b(x)}{2\sqrt{\varepsilon}},\ s(x) = \frac{f(x)}{2\sqrt{\varepsilon}} \tag{7}$$

The transition from equation (3) to equation (6) is admitted, because of the condition that ε is sufficiently small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Elsgolt's and Norkin [2].

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih$, $i = 0,1,2,..., N$.

Integrating equation (6) with respect to x from x_i to x_{i+1}, we get

$$y_{i+1} - y_i = q_{i+1} y(x_{i+1} - \delta) - q_i y(x_i - \delta) + \int_{x_i}^{x_{i+1}} \left[p(x) y(x - \sqrt{\varepsilon}) - q'(x) y(x - \delta) + r(x) y(x) + s(x) \right] dx$$

Where $y_i = y(x_i),\ p_i = p(x_i),\ q_i = q(x_i),\ r_i = r(x_i),\ s_i = s(x_i)$

By using Trapezoidal rule to evaluate the integral in the above equation, we get

$$\begin{aligned} y_{i+1} - y_i = {} & q_{i+1} y(x_{i+1} - \delta) - q_i y(x_i - \delta) + \frac{h}{2}\left[p_i y(x_i - \sqrt{\varepsilon}) + p_{i+1} y(x_{i+1} - \sqrt{\varepsilon}) \right] \\ & - \frac{h}{2}\left[q_i' y(x_i - \delta) + q_{i+1}' y(x_{i+1} - \delta) \right] + \frac{h}{2}\left[r_i y_i + r_{i+1} y_{i+1} + s_i + s_{i+1} \right] \end{aligned} \tag{8}$$

Again, by means of Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we get:

$$y(x_i - \sqrt{\varepsilon}) = y(x_i) - \sqrt{\varepsilon}\, y'(x_i) = y_i - \sqrt{\varepsilon}\left(\frac{y_i - y_{i-1}}{h} \right) = \left(1 - \frac{\sqrt{\varepsilon}}{h} \right) y_i + \frac{\sqrt{\varepsilon}}{h} y_{i-1} \tag{9a}$$

$$y(x_{i+1} - \sqrt{\varepsilon}) = y(x_{i+1}) - \sqrt{\varepsilon}\, y'(x_{i+1}) = y_{i+1} - \sqrt{\varepsilon}\left(\frac{y_{i+1} - y_i}{h} \right) = \left(1 - \frac{\sqrt{\varepsilon}}{h} \right) y_{i+1} + \frac{\sqrt{\varepsilon}}{h} y_i \tag{9b}$$

$$y(x_i - \delta) = y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_i - y_{i-1}}{h} \right) = \left(1 - \frac{\delta}{h} \right) y_i + \frac{\delta}{h} y_{i-1} \tag{9c}$$

$$y(x_{i+1} - \delta) = y(x_{i+1}) - \delta\, y'(x_{i+1}) = y_{i+1} - \delta\left(\frac{y_{i+1} - y_i}{h} \right) = \left(1 - \frac{\delta}{h} \right) y_{i+1} + \frac{\delta}{h} y_i \tag{9d}$$

By making use of equations (9) in (8), we obtain:

$$\begin{aligned} y_{i+1} - y_i = {} & \frac{\sqrt{\varepsilon}}{2} p_i y_{i-1} + \left[\frac{h}{2}\left(1 - \frac{\sqrt{\varepsilon}}{h} \right) p_i + \frac{\sqrt{\varepsilon}}{2} p_{i+1} \right] y_i + \frac{h}{2}\left(1 - \frac{\sqrt{\varepsilon}}{h} \right) p_{i+1} y_{i+1} \\ & + \left(1 - \frac{\delta}{h} \right)\left(q_{i+1} - \frac{h}{2} q_{i+1}' \right) y_{i+1} + \frac{\delta}{h}\left(q_{i+1} - \frac{h}{2} q_{i+1}' \right) y_i \\ & - \left(1 - \frac{\delta}{h} \right)\left(q_i + \frac{h}{2} q_i' \right) y_i - \frac{\delta}{h}\left(q_i + \frac{h}{2} q_i' \right) y_{i-1} + \frac{h}{2} r_i y_i + \frac{h}{2} r_{i+1} y_{i+1} \\ & + \frac{h}{2}(s_i + s_{i+1}) \end{aligned} \tag{10}$$

Rearranging equation (10), we get the following three term recurrence relation:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \text{, for } i = 1,2,..., N-1. \tag{11}$$

Where,

$$E_i = -\frac{\sqrt{\varepsilon}}{2} p_i + \frac{\delta}{h}\left(q_i + \frac{h}{2} q_i' \right)$$

$$F_i = 1 + \frac{h}{2} p_i - \frac{\sqrt{\varepsilon}}{2}(p_i - p_{i+1}) + \frac{\delta}{h}\left(q_{i+1} - \frac{h}{2} q_{i+1}' \right) - \left(1 - \frac{\delta}{h}\right)\left(q_i + \frac{h}{2} q_i' \right) + \frac{h}{2} r_i$$

$$G_i = 1 - \frac{h}{2}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right) p_{i+1} - \left(1 - \frac{\delta}{h}\right)\left(q_{i+1} - \frac{h}{2} q_{i+1}' \right) - \frac{h}{2} r_{i+1}$$

$$H_i = \frac{h}{2}(s_i + s_{i+1}) \tag{12}$$

We solve the above tri-diagonal system by using method of Discrete Invariant Imbedding Algorithm described in section 3.

2.2. Right-end boundary layer problems

We now assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of x = 1.

By using Taylor series expansion in the neighborhood of the point *x,* we have

$$y\left(x + \sqrt{\varepsilon}\right) = y(x) + \sqrt{\varepsilon}\, y'(x) + \frac{\varepsilon}{2} y''(x) \tag{13}$$

and consequently, equation (3) is replaced by the following first order differential equation:

$$y'(x) = p(x) y(x + \sqrt{\varepsilon}) + q(x) y'(x - \delta) + r(x) y + s(x) \tag{14}$$

for $0 \le \delta < 1$, where

$$p(x) = \frac{1}{\sqrt{\varepsilon}}, \quad q(x) = \frac{a(x)}{2\sqrt{\varepsilon}}, \quad r(x) = \frac{b(x) - 2}{2\sqrt{\varepsilon}}, \quad s(x) = \frac{-f(x)}{2\sqrt{\varepsilon}} \tag{15}$$

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih$, $i = 0,1,2,..., N.$ Integrating equation (14) with respect to x from x_{i-1} to x_i, we get

$$y_i - y_{i-1} = q_i y(x_i - \delta) - q_{i-1} y(x_{i-1} - \delta) + \int_{x_{i-1}}^{x_i} \left[p(x) y(x + \sqrt{\varepsilon}) - q'(x) y(x - \delta) + r(x) y(x) + s(x) \right] dx$$

By using Trapezoidal rule to evaluate the integral in the above equation, we get:

$$y_i - y_{i-1} = q_i y(x_i - \delta) - q_{i-1} y(x_{i-1} - \delta) + \frac{h}{2}\left[p_{i-1} y\left(x_{i-1} + \sqrt{\varepsilon}\right) + p_i y\left(x_i + \sqrt{\varepsilon}\right) \right]$$
$$- \frac{h}{2}\left[q_{i-1}' y(x_{i-1} - \delta) + q_i' y(x_i - \delta) \right] + \frac{h}{2}\left[r_{i-1} y_{i-1} + r_i y_i + s_{i-1} + s_i \right] \tag{16}$$

Again, by means of Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we get:

$$y(x_i-\delta)=y(x_i)-\delta y'(x_i)=y_i-\delta\left(\frac{y_{i+1}-y_i}{h}\right)=\left(1+\frac{\delta}{h}\right)y_i-\frac{\delta}{h}y_{i+1} \tag{17a}$$

$$y(x_{i-1}-\delta)=y(x_{i-1})-\delta\, y'(x_{i-1})=y_{i-1}-\delta\left(\frac{y_i-y_{i-1}}{h}\right)=\left(1+\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i \tag{17b}$$

$$y(x_i+\sqrt{\varepsilon})=y(x_i)+\sqrt{\varepsilon}\,y'(x_i)=y_i+\sqrt{\varepsilon}\left(\frac{y_{i+1}-y_i}{h}\right)=\left(1-\frac{\sqrt{\varepsilon}}{h}\right)y_i+\frac{\sqrt{\varepsilon}}{h}y_{i+1} \tag{17c}$$

$$y(x_{i-1}+\sqrt{\varepsilon})=y(x_{i-1})+\sqrt{\varepsilon}\,y'(x_{i-1})=y_{i-1}+\sqrt{\varepsilon}\left(\frac{y_i-y_{i-1}}{h}\right)=\left(1-\frac{\sqrt{\varepsilon}}{h}\right)y_{i-1}+\frac{\sqrt{\varepsilon}}{h}y_i \tag{17d}$$

By making use of equations (17) in (16), we obtain

$$\begin{aligned} y_i-y_{i-1}&=\frac{h}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)p_{i-1}y_{i-1}+\left[\frac{\sqrt{\varepsilon}}{2}p_{i-1}+\frac{h}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)p_i\right]y_i+\frac{\sqrt{\varepsilon}}{2}p_iy_{i+1}\\ &-\left(1+\frac{\delta}{h}\right)\left(q_{i-1}+\frac{h}{2}q'_{i-1}\right)y_{i-1}-\frac{\delta}{h}\left(q_i-\frac{h}{2}q'_i\right)y_{i+1}\\ &+\left(1+\frac{\delta}{h}\right)\left(q_i-\frac{h}{2}q'_i\right)y_i+\frac{\delta}{h}\left(q_{i-1}+\frac{h}{2}q'_{i-1}\right)y_i+\frac{h}{2}r_{i-1}y_{i-1}+\frac{h}{2}r_iy_i\\ &+\frac{h}{2}(s_{i-1}+s_i)\end{aligned} \tag{18}$$

Rearranging equation (18), we get the following three term recurrence relation:

$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i$, for $i=1,2,...,N-1$. (19)

Where,

$$\begin{aligned} E_i&=-1-\frac{h}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)p_{i-1}+\left(1+\frac{\delta}{h}\right)\left(q_{i-1}+\frac{h}{2}q'_{i-1}\right)-\frac{h}{2}r_{i-1}\\ F_i&=-1+\frac{h}{2}p_i+\frac{\sqrt{\varepsilon}}{2}(p_{i-1}-p_i)+\left(1+\frac{\delta}{h}\right)\left(q_i-\frac{h}{2}q'_i\right)+\frac{\delta}{h}\left(q_{i-1}+\frac{h}{2}q'_{i-1}\right)+\frac{h}{2}r_i\\ G_i&=-\frac{\sqrt{\varepsilon}}{2}p_i+\frac{\delta}{h}\left(q_i-\frac{h}{2}q'_i\right)\\ H_i&=\frac{h}{2}(s_{i-1}+s_i)\end{aligned} \tag{20}$$

We solve the above tri-diagonal system by using method of Discrete Invariant Imbedding Algorithm described in the next section.

3. Discrete invariant imbedding algorithm

We now describe the Thomas algorithm which is also called Discrete Invariant Imbedding Algorithm Angel & Bellman [1] to solve the three term recurrence relation:

$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i$, for $i=1,2,...,N-1$. (21)

Let us set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \quad , \text{for } i = N-2,\ N-1,...,2,1. \tag{22}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined. From (22), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{23}$$

Substituting (23) in (21), we have

$$y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \tag{24}$$

By comparing (22) and (24), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right) \tag{25}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \tag{26}$$

To solve these recurrence relations for $i = 2,3,..., N-1$, we need the initial conditions for W_0 and T_0. If we choose $W_0 = 0$, then we get $T_0 = \alpha$. With these initial values, we compute W_i and T_i for $i = 2,3,..., N-1$, from eqns.(25) and(26) in forward process, and then obtain y_i in the backward process from (22).

For further discussion on the conditions for the discrete invariant imbedding algorithm to be stable, one can see (Angel and Bellman [1], Elsgolt's and Norkin [2], and Kadalbajoo and Reddy [3]).

$$E_i > 0,\quad G_i > 0,\quad F_i \geq E_i + G_i \ \text{ and } \ |E_i| \leq |G_i|. \tag{27}$$

In our method, one can easily show that if the assumptions *a(x)* > 0, *b(x)* < 0 and $(\varepsilon - \delta a(x)) > 0$ hold, then the above conditions (27) hold and thus the invariant imbedding algorithm is stable.

4. Numerical experiments

To demonstrate the applicability of the method, two numerical experiments with left-end boundary layer and two numerical experiments with right-end boundary layer are considered. We compared the computed results with the exact solution of the problems where the exact solution of the problems is known; and we also have tested the effect of small delay parameter on computed solution of the problem for different values of δ of $o(\varepsilon)$ which are presented by numerical experiments whose exact solutions are not known.

Example 4.1. Consider the singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0; \quad x \in [0,1] \text{ with } y(0) = 1 \text{ and } y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{(1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}}$$

where $m_1 = \dfrac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$ and $m_2 = \dfrac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$.

The computational results are presented in the Table 1, 2, 3 and 4 for ε=0.01 and 0.001 for different values of δ.

Table 1. Numerical results of Example 4.1 for $\varepsilon = 0.01$, δ=0.001, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.04	0.3828886	0.3932546	1.037E-02
0.05	0.3867368	0.3923167	5.580E-03
0.06	0.3906236	0.3946420	4.018E-03
0.07	0.3945495	0.3980572	3.508E-03
0.09	0.4025200	0.4058020	3.282E-03
0.20	0.4493248	0.4493248	0.000E+00
0.40	0.5488078	0.5488078	0.000E+00
0.60	0.6703169	0.6703169	0.000E+00
0.80	0.8187288	0.8187288	0.000E+00
1.00	1.0000000	1.0000000	0.000E+00

Table 2. Numerical results of Example 4.1 for $\varepsilon = 0.01$, δ=0.003, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.03	0.3685689	0.3900410	2.147E-02
0.04	0.3845921	0.3874350	2.843E-03
0.05	0.3853987	0.3897590	4.361E-03
0.07	0.3937812	0.3971220	3.341E-03
0.09	0.4017884	0.4050630	3.274E-03
0.30	0.4958891	0.4989910	3.102E-03
0.50	0.6059232	0.6086280	2.705E-03
0.70	0.7403729	0.7423540	1.981E-03
0.90	0.9046561	0.9054620	8.062E-04
1.00	1.0000000	1.0000000	0.000E+00

Table 3. Numerical results of Example 4.1 for $\varepsilon = 0.001$, δ =0.0003, N =100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.02	0.3753151	0.3755683	2.532E-04
0.03	0.3790854	0.3793402	2.548E-04
0.05	0.3867434	0.3869979	2.545E-04
0.07	0.3945560	0.3948103	2.543E-04
0.08	0.3985214	0.3987754	2.540E-04
0.20	0.4493313	0.4495803	2.490E-04
0.40	0.5488138	0.5490419	2.281E-04
0.60	0.6703218	0.6705075	1.857E-04
0.80	0.8187319	0.8188452	1.133E-04
1.00	1.0000000	1.0000000	0.000E+00

Table 4. Numerical results of Example 4.1 for $\varepsilon = 0.001$, δ =0.0008, N =100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.02	0.3767505	0.3753847	1.366E-03
0.03	0.3788219	0.3791566	3.347E-04
0.04	0.3827179	0.3829664	2.485E-04
0.06	0.3904489	0.3907013	2.524E-04
0.08	0.3983404	0.3985924	2.520E-04
0.20	0.4491538	0.4494008	2.470E-04
0.40	0.5486512	0.5488775	2.263E-04
0.60	0.6701894	0.6703736	1.842E-04
0.80	0.8186510	0.8187635	1.125E-04
1.00	1.0000000	1.0000000	0.000E+00

Example 4.2. Now we consider an example of variable coefficient singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0 \quad \text{with } y(0) = 1,\ y(1) = 1$$

For which the exact solution is not known. This example is considered to show the effect of the small shift on the boundary layer solution.

The computational results are presented in the Table 5 and 6 for ε=0.01 and 0.001 for different values of δ.

Table 5. Numerical results of example 4.2 for ε=0.01, N=100 different values of δ

	Numerical Solutions				
x	δ=0.00	δ=0.001	δ=0.002	δ=0.003	δ=0.004
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.03	0.2830336	0.2818345	0.2802429	0.2695371	0.2239334
0.05	0.2882965	0.2876440	0.2869820	0.2856769	0.2756947
0.07	0.2942855	0.2936334	0.2929779	0.292280	0.2898927
0.09	0.3004657	0.2998095	0.2991498	0.2984845	0.2975120
0.10	0.3036281	0.3029698	0.3023080	0.3016434	0.3011034
0.20	0.3381277	0.3374511	0.3367710	0.3360870	0.3353999
0.40	0.4264295	0.4257381	0.4250430	0.4243436	0.4236403
0.60	0.5510259	0.5503808	0.5497320	0.5490786	0.5484208
0.80	0.7313973	0.7309322	0.7304640	0.7299923	0.7295169
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6. Numerical results of example 4.2 for ε=0.001, N=100 different values of δ

	Numerical Solutions				
x	δ=0.0001	δ=0.0002	δ=0.0003	δ=0.0004	δ=0.0008
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.2793129	0.2790174	0.2788556	0.2788358	0.2803663
0.04	0.2847015	0.2846368	0.284572	0.2845069	0.2842517
0.06	0.2905979	0.2905328	0.2904677	0.2904021	0.2901406
0.08	0.2966775	0.2966121	0.2965465	0.2964805	0.2962173
0.10	0.3029474	0.3028815	0.3028156	0.3027491	0.3024842
0.20	0.3374169	0.3373493	0.3372816	0.3372131	0.3369409
0.40	0.4256782	0.4256091	0.4255397	0.4254698	0.4251915
0.60	0.5502985	0.5502338	0.5501689	0.5501038	0.5498438
0.80	0.7308509	0.7308043	0.7307575	0.7307104	0.7305229
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Example 4. 3. Consider the singularly perturbed delay differential equation with right layer:

$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0$; $x \in [0,1]$ with $y(0) = 1$ and $y(1) = -1$.

The exact solution is given by $y(x) = \dfrac{(1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x}}{e^{m_2} - e^{m_1}}$

Where $m_1 = \dfrac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$ and $m_2 = \dfrac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$.

The computational results are presented in the Table 7, 8 and 9 for ε=0.01 and different values of δ .

Table 7. Numerical results of example 4.3 for ε=0.01, $\delta = 0.002$, N = 100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.10	0.9051042	0.9058985	7.943E-04
0.30	0.7414737	0.7434276	1.954E-03
0.50	0.6074253	0.6100955	2.670E-03
0.80	0.4503898	0.4535618	3.172E-03
0.91	0.4036031	0.4061460	2.543E-03
0.93	0.3955912	0.3951282	4.630E-04
0.94	0.3915125	0.3862416	5.271E-03
0.95	0.3869567	0.3708223	1.613E-02
0.96	0.3801894	0.3401704	4.002E-02
1.00	-1.0000000	-1.0000000	0.000E+00

Table 8. Numerical results of example 4.3 for ε=0.01, $\delta = 0.003$, N = 100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.10	0.9051947	0.9059848	7.901E-04
0.30	0.7416958	0.7436400	1.944E-03
0.60	0.5501123	0.5530005	2.888E-03
0.80	0.4507494	0.4539073	3.158E-03
0.91	0.4039555	0.4059563	2.001E-03
0.93	0.3958005	0.3933562	2.444E-03
0.94	0.3913622	0.3825042	8.858E-03
0.95	0.3856679	0.3635170	2.215E-02
0.97	0.3492044	0.2511996	9.800E-02
1.00	-1.0000000	-1.0000000	0.000E+00

Table 9. Numerical results of example 4.3 for ε=0.01, $\delta = 0.008$, N = 100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000E+00
0.20	0.8201722	0.8215815	1.409E-03
0.40	0.6726823	0.6749963	2.314E-03
0.60	0.5517152	0.5545645	2.849E-03
0.80	0.4525010	0.4556031	3.102E-03
0.91	0.4042167	0.4004646	3.752E-03
0.92	0.3984778	0.3900229	8.455E-03

0.93	0.3908216	0.3747227	1.610E-02
0.94	0.3790436	0.3508411	2.820E-02
1.00	-1.0000000	-1.0000000	0.000E+00

Example 4.4. Now we consider an example of variable coefficient singularly perturbed delay differential equation with right layer:

$$\varepsilon y''(x) - e^{x} y'(x-\delta) - y(x) = 0 \text{ with } y(0)=1,\ y(1)=1$$

For which the exact solution is not known. This example is considered to show the effect of the small shift on the boundary layer solution.
The computational results are presented in the Table 10 and 11 for ε=0.01 and 0.001 for different values of δ.

Table 10. Numerical results of example 4.4 for ε =0.01, N =100 different values of δ

x	Numerical solutions			
	$\delta = 0.00$	$\delta = 0.003$	$\delta = 0.006$	$\delta = 0.008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8344690	0.8353115	0.8361393	0.8366833
0.40	0.7195132	0.7207859	0.7220381	0.7228627
0.60	0.6372644	0.6387585	0.6402303	0.6412004
0.80	0.5769618	0.5785688	0.5801532	0.5811983
0.91	0.5505952	0.5522395	0.5539518	0.5553138
0.93	0.5462266	0.5479044	0.5501223	0.5524401
0.94	0.5440876	0.5458574	0.5489668	0.5525444
0.95	0.5419781	0.5441055	0.5493786	0.5553575
0.97	0.5378680	0.5469612	0.5683851	0.5858082
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 11. Numerical results of example 4.4 for ε =0.001, N =100 different values of δ

x	Numerical Solutions			
	$\delta = 0.0001$	$\delta = 0.0003$	$\delta = 0.0006$	$\delta = 0.0008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8343205	0.8343771	0.8344627	0.8345196
0.30	0.7718209	0.7718945	0.7720051	0.7720788
0.50	0.6748798	0.6749741	0.6751158	0.6752099
0.60	0.6370457	0.6371462	0.6372970	0.6373973
0.90	0.5526100	0.5527205	0.5528854	0.5529957
0.92	0.5481805	0.5482913	0.5484567	0.5485674
0.94	0.5438731	0.5439842	0.5441500	0.5442612
0.96	0.5396839	0.5397965	0.5399709	0.5400962
0.98	0.5358225	0.5364740	0.5379835	0.5392989
1.00	1.0000000	1.0000000	1.0000000	1.0000000

5. Discussions and conclusions

We have presented a numerical integration method to solve singularly perturbed delay differential equations. The scheme is repeated for different choices of the delay parameter, δ and perturbation parameter, ε. The choice of δ is not unique, but can assume any number of values satisfying the condition, 0< δ <<1 and $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = O(1)$ and τ is not too large, Lange and Miura [9]. To reduce the amount of computations, we fix the mesh size *h* and vary the value of δ. Although the solutions are computed at all the points of mesh size *h*, only few values have been reported. To demonstrate the efficiency of the method, we considered some numerical examples of boundary value problems with constant and variable coefficients for different values of the delay and perturbation parameters. Most existing numerical methods produce good results for $h \le \varepsilon$ which is very costly and time consuming (Example see [7]) whereas, in this paper we are tried to develop the method which produce good results for $h \ge \varepsilon$; and from the computational results, it is observed that the proposed method approximates the exact solution very well for $\varepsilon \le h$ for which other classical finite difference methods fails to give good results. The small shift, δ affects both the boundary layer solutions (left and right) in similar fashion but reversely. That is, as δ increases the size/thickness of the left boundary layer decreases while that of the right boundary layer increases (See Table 5, 6, 10, 11). This method does not depend on asymptotic expansion as well as on the matching of the coefficients. Thus, we have devised an alternative technique of solving boundary value problems for singularly perturbed delay differential equations, which is easily implemented on computer and is also practical.

References

[1] Angel, E. and Bellman, R. 1972. "*Dynamic Programming and Partial differential equations*". Academic Press. New York.
[2] Elsgolt's, L. E. and Norkin, S. B. 1973. "*Introduction to the Theory and Applications of Differential Equations with Deviating Arguments*". Academic Press. New York.
[3] Kadalbajoo, M. K. and Reddy Y. N. 1986. A non-asymptotic method for general linear singular perturbation problems. *Journal of Optimization Theory and Applications*, 55: 256-269.
[4] Kadalbajoo, M. K. and Sharma, K. K. 2008. A numerical method based on finite difference for Boundary value problems for singularly perturbed delay differential equations. *Applied Mathematics and Computation*, 197: 692-707.
[5] Kadalbajoo, M. K., Patidar, K. C., and Sharma, K. K. 2006. e-Uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general DDEs. *Applied Mathematics and Computation*, 182: 119-139.
[6] Kadalbajoo, M. K. and Sharma, K. K. 2004. Numerical Analysis of Boundary Value Problems for Singularly Perturbed Differential-Difference Equations: Small Shifts of Mixed Type with Rapid Oscillations. *Comm. Numer. Meth. Eng*. 20: 167-182.
[7] Kadalbajoo, M. K. and Sharma, K. K. 2004. Numerical Analysis of Singularly Perturbed Delay Differential Equations with Layer Behavior. *Appl.Math. Comput*, 157, 1: 11-28.
[8] Kuang, Y. 1993. "*Delay Differential Equations with Applications in Population Dynamics*". Academic Press.
[9] Lange, C. G. and Miura, R. M. 1994. Singular Perturbation Analysis of Boundary-Value

Problems for Differential Difference Equations. V. Small Shifts with Layer Behavior. *SIAM Journal on Applied Mathematics,* 54: 249-272.

[10] Lange, C. G. and Miura, R. M. 1994. Singular Perturbation Analysis of Boundary-Value Problems for Differential Difference Equations. VI. Small Shifts with Rapid Oscillations. *SIAM Journal on Applied Mathematics*, 54: 273-283.

[11] Longtin, A. and Milton, J. 1988. Complex oscillations in the human pupil light reflex with mixed and delayed feedback. *Mathematical Biosciences*, 90: 183-199.

[12] Patidar, K. C. and Sharma, K. K. 2006. ε-Uniformly Convergent Non-Standard Finite Difference Methods for Singularly Perturbed Differential Difference Equations with Small Delay. *Applied Mathematics and Computation*, 175: 864-890.

[13] Phaneendra, K., Reddy, Y. N., and Soujanya, GBSL. 2011. A Seventh Order Numerical Method for Singular Perturbed Differential-Difference Equations with Negative Shift. *Nonlinear Analysis: Modeling and Control*, 16, 2: 206-219.

[14] Rao, R. N. and Chakravarthy, P. P. 2011. A Fourth Order Finite Difference Method for Singularly Perturbed Differential-difference Equations. *American Journal of Computational and Applied Mathematics*, 1, 1: 5-10.

[15] Yadaw, A. S. and Kadalbajoo, M. K. 2009. Parameter-Uniform Ritz-Galerkin Finite Element Method for Singularly Perturbed Delay Differential Equations with Delay in Convection Term. *International Journal of Pure and Applied Mathematics*, 57, 4: 459-474.