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Parameter Fitted Scheme for Singularly Perturbed Delay Differential Equations

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Abstract: In this paper, we presented a parameter fitted scheme to solve singularly perturbed delay differential equations of second order with left and right boundary. In this technique, approximating the term containing negative shift by Taylor series, we modified the singularly perturbed delay differential equations. We introduced a fitting parameter on the highest order derivative term of the modified problem. The fitting parameter is to be determined from the scheme using the theory of singular Perturbation. Finally, we obtained a three term recurrence relation that can be solved using Thomas algorithm. The applicability of the method is tested by considering four linear problems (two problems on left layer and two problems on right layer). It is observed that when the delay parameter is smaller than the perturbation parameter, the layer behavior is maintained.

Keywords: Delay differential equations; singular perturbation; parameter fitted.

1. Introduction

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. In recent papers [9-13] the terms negative or left shift and positive or right shift have been used for delay and advance respectively. The smoothness of the solutions of such singularly perturbed differential-difference equation deteriorates when the parameter tends to zero. Such problems are found throughout the literature on epidemics and population dynamics where these small shifts play an important role in the modeling of various real life phenomena [18]. Boundary value problems in differential-difference equations arise in a very natural way in studying variation problems in control theory where the problem is complicated by the effect of time delays in signal transmission [7].

The differential-difference equation plays an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. For a detailed discussion on differential-difference equation one may refer to the books and high level monographs: Bellen[2], Driver[16], Bellman and Cooke [17].

Lange and Miura [5] gave an asymptotic approach in the study of a class of boundary value problems for linear second order differential-difference equations in which the highest order

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derivative is multiplied by a small parameter. It has been shown that the layer behavior can change its character and even be destroyed as the shifts increase but remain small. In [7], similar boundary value problems with solutions that exhibit rapid oscillations are studied. Based on finite difference scheme, fitted mess and B-spline technique, piecewise uniform mesh an extensive numerical work had been initiated by M. K. Kadalbajoo and K. K. Sharma in their papers [9-14] for solving singularly perturbed delay differential equations. In [8], M. Gulsu presented matrix methods for approximate solution of the second order singularly perturbed delay differential equations.

It is well known that the classical methods fail to provide reliable numerical results for such problems (in the sense that the parameter and the mess size cannot vary independently). Lange and Miura[3-5] gave asymptotic approaches in the study of class of boundary value problems for linear second order differential difference equations in which the highest order derivative is multiplied by small parameter.

The aim of this paper is to provide a parameter fitted scheme to solve singularly perturbed delay differential equations of second order with left or right boundary. The effect of small shifts on the boundary layer solution of the problem has been considered. In this technique, by approximating the term containing negative shift by Taylor series expansion, we modify the singularly perturbed delay differential equations. We introduce a fitting parameter on the highest order derivative term of the new equation. The fitting parameter is to be determined from the upwind scheme using the theory of singular Perturbation. We obtain a three term recurrence relation that can be solved using Thomas algorithm. The applicability of the method is tested by considering four problems which have been widely discussed in literature (two linear problems on left layer, two linear problems on right layer. It is observed that when the delay parameter is smaller than the perturbation parameter, the layer behavior is maintained.

2. The parameter fitted scheme

To describe the method, we first consider a linear singularly perturbed delay differential two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{1}$$

with $y(0) = \phi(x); \ -\delta \le x \le 0$ (1a)

and $y(1) = \beta$ (1b)

where ε is a small positive parameter (0<ε<<1), $b(x), f(x)$ are bounded functions in $(0,1)$ and α, β are known constants. Furthermore, we assume that $a(x) \ge M > 0$ throughout the interval [0,1], where M is a positive constant. Under these assumptions, (1) has a unique solution y(x) which in general, displays a boundary layer of width O(ε) at $x = 0$.

Assuming $\phi(0) = \alpha; \ -\delta \le x \le 0$, we can rewrite (1) in the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{2}$$

with $y(0) = \alpha; \ -\delta \le x \le 0$ (2a)

and $y(1) = \beta$ (2b)

Approximating $y'(x-\delta)$ by the linear interpolation, we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{3}$$

Substituting equation (3) in to equation (2), we get

$$(\varepsilon - \delta a(x) y''(x) + a(x) y'(x) + b(x) y(x) = f(x) \tag{4}$$

For Possible choices of δ such that $0 \le \gamma = \varepsilon - \delta a(x) << 1$ (here γ varies as ε, δ and $\alpha(x)$ vary), from the theory of singular perturbations it is known that the solution of (4) and (3) is of the form [O' Malley [15]; pp 22-26].

$$y(x) = y_0(x) + \frac{a(0)}{a(x)} (\alpha - y_0(0)) e^{-\int_0^x \left(\frac{a(x)}{\gamma} - \frac{b(x)}{a(x)} \right) dx} + O(\varepsilon) \tag{5}$$

where $y_0(x)$ is the solution of

$a(x)y_0'(x) + b(x)y_0(x) = f(x)$, $y_0(1) = \beta$.

By taking the Taylor's series expansion for a(x) and b(x) about the point '0' and restricting to their first terms, (5) becomes,

$$y(x) \simeq y_0(x) + (\alpha - y_0(0)) e^{-\left(\frac{a(0)}{\gamma} - \frac{b(0)}{a(0)} \right) x} + O(\varepsilon) \tag{6}$$

Now we divide the interval [0, 1] into N equal parts with constant mesh length h. Let $0 = x_0, x_1, ..., x_N = 1$ be the mesh points. Then we have $x_i = ih; i = 0,1,...,N$.

From (6) we have

$$y(ih) = y_0(x) + (\alpha - y_0(0)) e^{-\left(\frac{a^2(0) - \gamma b(0))}{\gamma a(0)} \right) ih} \tag{7}$$

and $$\lim_{h \to 0} y(ih) = y_0(0) + (\alpha - y_0(0)) e^{-\left(\frac{a^2(0) - \gamma b(0)}{a(0)} \right) i\rho} \tag{8}$$

where $\rho = \dfrac{h}{\gamma}$

The upwind scheme corresponding to equation (4) is

$$\frac{\gamma}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_i}{h}(y_{i+1} - y_i) + b_i y_i = f_i \quad ; \quad i = 1,2,...N-1 \tag{9}$$

where $a(x_i) = a_i; b(x_i) = b_i; f(x_i) = f_i; y(x_i) = y_i$.

Now, we introduce a fitting factor $\sigma(\rho)$ in the above scheme (9) as follows

$$\frac{\sigma(\rho)\gamma}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_i}{h}(y_{i+1} - y_i) + b_i y_i = f_i \quad ; \quad i = 1,2,...N-1 \tag{10}$$

with y (0)=α and y (1)= β.

The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (10) converges uniformly to the solution of (1)-(2).

Multiplying (10) by h and taking the limit as $h \to 0$; we get

$$\lim_{h \to 0} \left[\left(\frac{\sigma}{\rho}\right) \big(y(ih+h) - 2y(ih) + y(ih-h) \big) + a(ih) \big(y(ih+h) - y(ih) \big) \right] = 0 \tag{11}$$

By substituting (5) in (11) and simplifying, we get the constant fitting factor

$$\sigma = \frac{\rho a(0)}{4} \quad \frac{[1-e^{-(\frac{a^2(0)-\gamma b(0)}{a(0)})}]}{[\sinh((\frac{a^2(0)-\gamma b(0)}{a(0)})/2)]^2} \tag{12}$$

The equivalent three term recurrence relation of equation (10) is given by:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; \quad i=1,2,3, \ldots\ldots,N-1 \tag{13}$$

where; $E_i = \frac{\sigma\gamma}{h^2}; F_i = \frac{2\sigma\gamma}{h^2} + \frac{a_i}{h} - b_i; G_i = \frac{\sigma\gamma}{h^2} + \frac{a_i}{h}; H_i = f_i$

This gives us the tri diagonal system which can be solved easily by Thomas Algorithm.

Thomas Algorithm

A brief discussion on solving the three term recurrence relation using Thomas algorithm which also called *Discrete Invariant Imbedding* (Angel & Bellman [6])is presented as follows:

Consider the scheme given in (13):

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; \quad i=1,2,3, \ldots\ldots,N-1$ subject to the boundary conditions

$$y_0 = y(0) = \alpha ; \quad \text{and} \quad y_N = y(1) = \beta \tag{13a}$$

We set $y_i = W_i y_{i+1} + T_i$ for i = N-1, N-2, …. 2, 1. (13b)

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (13b), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{13c}$$

By substituting (13c) in (13), we get $E_i(W_{i-1}y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i$.

$$\therefore \quad y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right) \tag{13d}$$

By comparing (13d) and (13b), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) \tag{13e}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{13f}$$

To solve these recurrence relations for i=0,1,2,3,……,N-1, we need the initial conditions for W_0 and T_0. For this we take $y_0 = \alpha = W_0 y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \alpha$. With these initial values, we compute W_i and T_i for i=1,2,3,….,N-1 from (13e-13f) in forward process, and then obtain y_i in the backward process from (13b)and (13a).

The conditions for the discrete invariant embedding algorithm to be stable are (see [1-2]):

$E_i > 0, F_i > 0$, $F_i \geq E_i + G_i$ and $|E_i| \leq |G_i|$ (13g)

In this method, if the assumptions $a(x) > 0, b(x) < 0$ and $(\varepsilon - \delta a(x)) > 0$ hold, one can easily show that the conditions given in (13g) hold and thus the invariant imbedding algorithm is stable.

3. Right-end boundary layer

We now assume that $a(x) \le M < 0$ throughout the interval [0, 1], where M is some negative constant. This assumption merely implies that the boundary layer for equation (1)-(2) will be in the neighborhood of x=1. From the theory of singular perturbations it is known that the solution of (4) and (2) is of the form [cf. O' Malley [15]; pp 22-26].

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))\, e^{\int_x^1 \left(\frac{a(x)}{\varepsilon - \delta a(x)} - \frac{b(x)}{a(x)}\right) dx} + O(\varepsilon) \tag{14}$$

where $y_0(x)$ is the solution of $a(x)y_0'(x) + b(x)y_0(x) = f(x)$, $y_0(0) = \alpha$.

Similar to the left layer problems, for possible choices of δ such that $0 \le \gamma = \varepsilon - \delta a(x) << 1$, we have

$$y(x) = y_0(x) + (\beta - y_0(1))\, e^{\left(\frac{a(1)}{\gamma} - \frac{b(1)}{a(1)}\right)(1-x)} + O(\varepsilon) \tag{15}$$

From (15) we have

$$\lim_{h \to 0} y(ih) = y_0(0) + (\beta - y_0(1))\, e^{\left(\frac{a^2(1) - \gamma b(1)}{a(1)}\right)\left(\frac{1}{\gamma} - i\rho\right)} \tag{16}$$

where, $\rho = \frac{h}{\gamma}$,

We introduce a fitting factor $\sigma(\rho)$ in the scheme corresponding to equation (4)

$$\frac{\sigma(\rho)\gamma}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_i}{h}(y_i - y_{i-1}) + b_i y_i = f_i \quad ; 1 \le i \le N-1 \tag{17}$$

with $y(0)=\alpha$ and $y(1)=\beta$.

Multiplying (17) by h and taking the limit as $h \to 0$; we get the value of the fitting factor:

$$\sigma = -\frac{\rho a(0)}{4} \quad \frac{[1 - e^{\left(\frac{a^2(1) - \gamma b(1)}{a(1)}\right)}]}{[\sinh((\frac{a^2(1) - \gamma b(1)}{a(1)})/2)]^2} \tag{18}$$

The equivalent three term recurrence relation of equation (17) is given by:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; \quad i=1,2,3, \ldots, N-1 \tag{19}$$

where; $E_i = \frac{\sigma\gamma}{h^2} - \frac{a_i}{h}$; $F_i = \frac{2\sigma\gamma}{h^2} - \frac{a_i}{h} - b_i$; $G_i = \frac{\sigma\gamma}{h^2}$; $H_i = f_i$ This gives us the tri diagonal system which can be solved easily by Thomas Algorithm.

Description of the Method

To solve problems of the form given in equation (1)-(2), we followed the procedure:

Step 1. Modify the Singularly Perturbed Delay Differential Equation (1) to the form (4).

Step 2. Introduce the fitting parameter in (4) and determine its value using the theory of singular perturbation.

Step 3. Solve the tri-diagonal system (13) and (19) with the boundary conditions (2) using Thomas Algorithm by taking different values of the perturbation and the delay parameter.

4. Numerical examples

To demonstrate the applicability of the method, we considered four numerical experiments (two problems with left-end and two with right-end boundary layer). We presented the absolute error compared to the exact solution of the problems. For the examples not having the exact solution, the absolute error is calculated using the double mesh principle.

Example 1.

Consider a singularly perturbed delay differential equation with left layer:

$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0; \quad 0 \in [0,1]$ with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by: $y(x) = \dfrac{[(1-e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}}$

where, $m_1 = \dfrac{\left(-1-\sqrt{1+4(\varepsilon-\delta)}\right)}{2(\varepsilon-\delta)}$ and $m_2 = \dfrac{\left(-1+\sqrt{1+4(\varepsilon-\delta)}\right)}{2(\varepsilon-\delta)}$.

Example 2.

Now we consider an example of variable coefficient singularly perturbed delay differential equation with left layer:

$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0; \quad 0 \in [0,1]$ with $y(0) = 1$ and $y(1) = 1$.

Example 3.

Consider a singularly perturbed delay differential equation with left layer:

$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0; \quad 0 \in [0,1]$ with $y(0) = 1$ and $y(1) = -1$.

The exact solution is given by: $y(x) = \dfrac{[(1+e^{m_2})e^{m_1 x} - (e^{m_1} + 1)e^{m_2 x}}{e^{m2} - e^{m_1}}$

where, $m_1 = \dfrac{\left(1-\sqrt{1+4(\varepsilon+\delta)}\right)}{2(\varepsilon+\delta)}$ and $m_2 = \dfrac{\left(1+\sqrt{1+4(\varepsilon+\delta)}\right)}{2(\varepsilon+\delta)}$.

Example 4.

Now we consider an example of variable coefficient singularly perturbed delay differential equation with right layer:

$\varepsilon y''(x) - e^{x} y'(x-\delta) - xy(x) = 0; \quad 0 \in [0,1]$ with $y(0) = 1$ and $y(1) = 1$.

5. Discussion and conclusions

We presented a parameter fitted scheme to solve singularly perturbed delay differential equations of second order with left and right boundary. Approximating the term containing negative shift by Taylor series, we modified the singularly perturbed delay differential equations. We introduced a variable fitting parameter on the highest order derivative term of the new equation and determined its value using the theory of singular Perturbation; O'Malley [15]. We solved the resulting three term recurrence relation using Thomas algorithm.

To test the applicability of the new method, four problems were considered by taking different values of the delay parameter δ , the perturbation parameter ε and the same mesh size. We considered δ to increase from 0.1ε to 0.9ε. For left layer problems, it is observed that as δ increase from 0.1ε to 0.9ε, ϒ decreases and absolute maximum error decreases for the constant coefficient problem and increases for the variable coefficient problem. For right layer problems, it is observed that as δ increase from 0.1ε to 0.9ε, ϒ increases and absolute maximum error increases for the constant coefficient problem and decreases for the variable coefficients problem. Overall, it is observed that when the delay parameter is smaller than the perturbation parameter, the layer behavior is maintained.

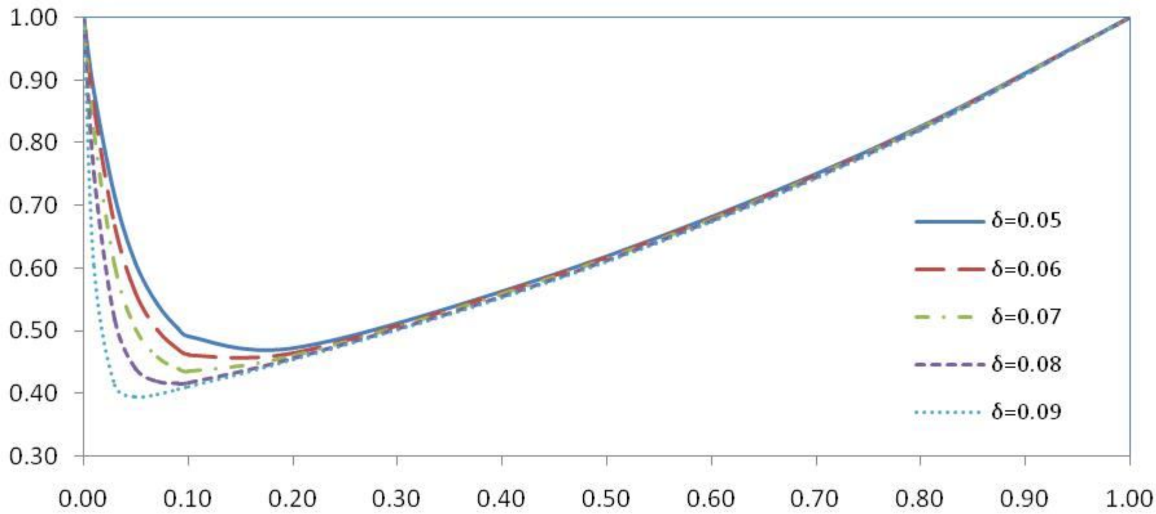

Figure 1. Graph for solution of example 1. for ε=0.1 and different values of δ

Table 1. Results for example 1. for ε=0.1, h=0.01

x(i)	δ=0.05	δ=0.06	δ=0.07	δ=0.08	δ=0.09
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	0.88232340	0.85747830	0.81874800	0.75035960	0.60202650
0.03	0.71413730	0.66804750	0.60503200	0.51764120	0.41243570
0.05	0.60853200	0.56084810	0.50349550	0.44035200	0.39445670
0.07	0.54332050	0.50166710	0.45740380	0.41812020	0.39893030
0.09	0.50420220	0.47055380	0.43875630	0.41545710	0.40646000
0.10	0.49140340	0.46166860	0.43507910	0.41713960	0.41045980
0.20	0.47294680	0.46525620	0.45996560	0.45633130	0.45312560
0.40	0.56349150	0.56079290	0.55804090	0.55519590	0.55228590
0.60	0.68214510	0.68003000	0.67781370	0.67550840	0.67314600
0.80	0.82591970	0.82463920	0.82329440	0.82189320	0.82045470
1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
Abs. Max. Err.	1.10E-02	8.87E-03	6.70E-03	4.45E-03	2.11E-03

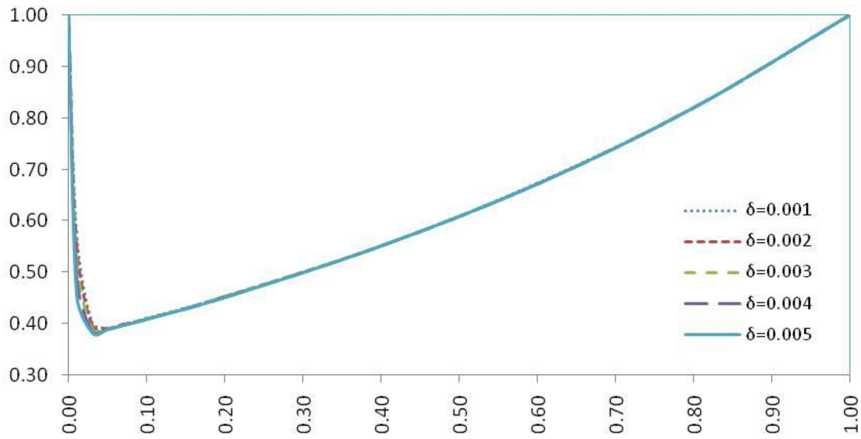

Figure 2. Graph for solution of example 1. for ε=0.01 and different values of δ

Table 2. Results for example 1. for ε=0.01,h=0.01

x(i)	δ=0.001	δ=0.002	δ=0.003	δ=0.004	δ=0.005
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	0.57798310	0.55145200	0.52233970	0.49078280	0.45750990
0.03	0.40377390	0.39626760	0.39019620	0.38573960	0.38294290
0.05	0.39250510	0.39108960	0.39014440	0.38953700	0.38915430
0.07	0.39834220	0.39788560	0.39752770	0.39721380	0.39693830
0.09	0.40610100	0.40577030	0.40546940	0.40517500	0.40490410
0.10	0.41012630	0.40980730	0.40951060	0.40921740	0.40894700
0.20	0.45281370	0.45250630	0.45221660	0.45192920	0.45166360
0.40	0.55200070	0.55171970	0.55145470	0.55119180	0.55094890
0.60	0.67291420	0.67268570	0.67247050	0.67225670	0.67205920
0.80	0.82031350	0.82017420	0.82004300	0.81991260	0.81979220
1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
Abs. Max. Err.	1.83E-03	1.53E-03	1.17E-03	7.83E-04	5.64E-04

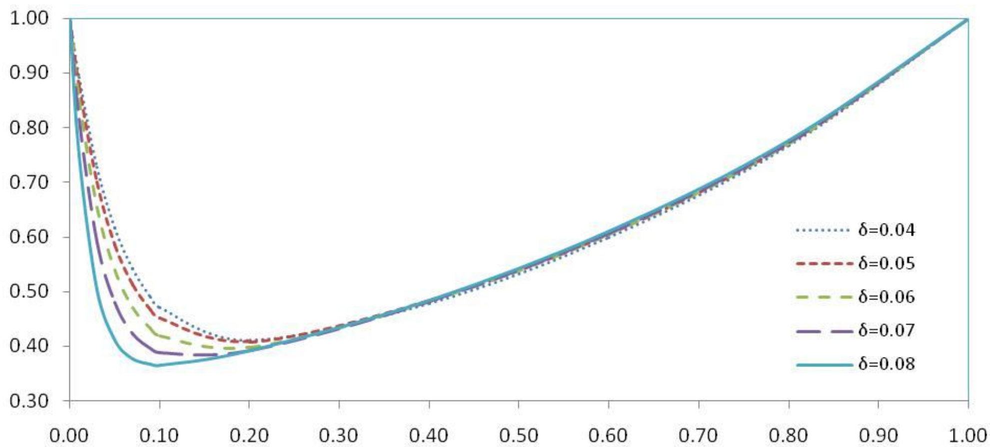

Figure 3. Graph for solution of example 2. for ε=0.1 and different values of δ

Table 3. Results for example 2. for (ε=0.1)

x(i)	δ=0.04	δ=0.05	δ=0.06	δ=0.07	δ=0.08
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	0.89142690	0.87695430	0.85244070	0.81439880	0.74754270
0.03	0.72769140	0.69983490	0.65426640	0.59172180	0.50355450
0.05	0.61609390	0.58623670	0.53843260	0.47979630	0.41184090
0.07	0.54031970	0.51343870	0.47066320	0.42334380	0.37741770
0.09	0.48930470	0.46715350	0.43146840	0.39558160	0.36606260
0.10	0.47062210	0.45097150	0.41885290	0.38797470	0.36454320
0.20	0.41023060	0.40794490	0.39830550	0.39203740	0.39085780
0.40	0.47800690	0.48306830	0.48163960	0.48164770	0.48454350
0.60	0.59904760	0.60513100	0.60488540	0.60592000	0.60985720
0.80	0.76655330	0.77165180	0.77193620	0.77323830	0.77683300
1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
Abs. Max. Err.	6.29E-04	1.26E-03	1.55E-03	2.00E-03	2.77E-03

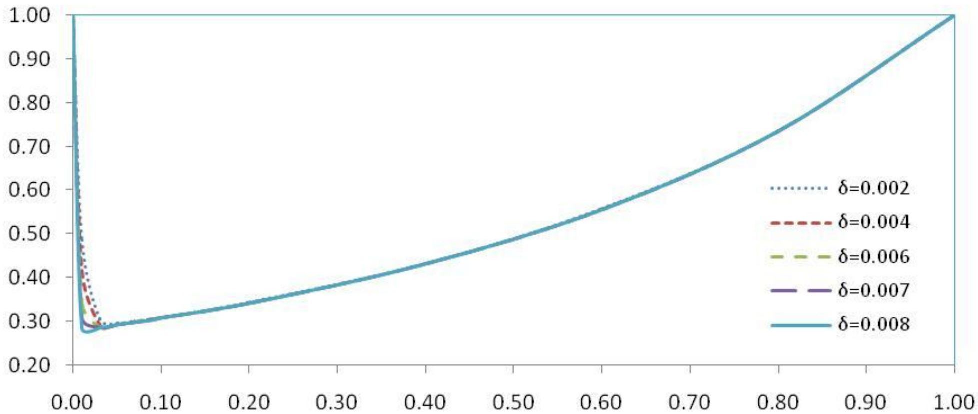

Figure 4. Graph for solution of example 2. for ε=0.01 and different values of δ

Table 4. Results for example 2. for (ε=0.01)

x(i)	δ=0.002	δ=0.004	δ=0.006	δ=0.007	δ=0.008
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.01	0.48692950	0.41688970	0.34148500	0.30790000	0.28638180
0.03	0.30546100	0.29240020	0.28767130	0.28693920	0.28667950
0.05	0.29608490	0.29356830	0.29308950	0.29276150	0.29254270
0.07	0.30086480	0.29943280	0.29913230	0.29880640	0.29858650
0.09	0.30699360	0.30565200	0.30536520	0.30503800	0.30481720
0.10	0.31017690	0.30883670	0.30855370	0.30822590	0.30800480
0.20	0.34493740	0.34354730	0.34330790	0.34297780	0.34275800
0.40	0.43357130	0.43211600	0.43200220	0.43169310	0.43150900
0.60	0.55788340	0.55648930	0.55654050	0.55629860	0.55620170
0.80	0.73649640	0.73546280	0.73564090	0.73551610	0.73553140
1.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
Abs. Max. Err.	2.69E-04	2.00E-04	5.41E-04	7.48E-04	1.17E-03

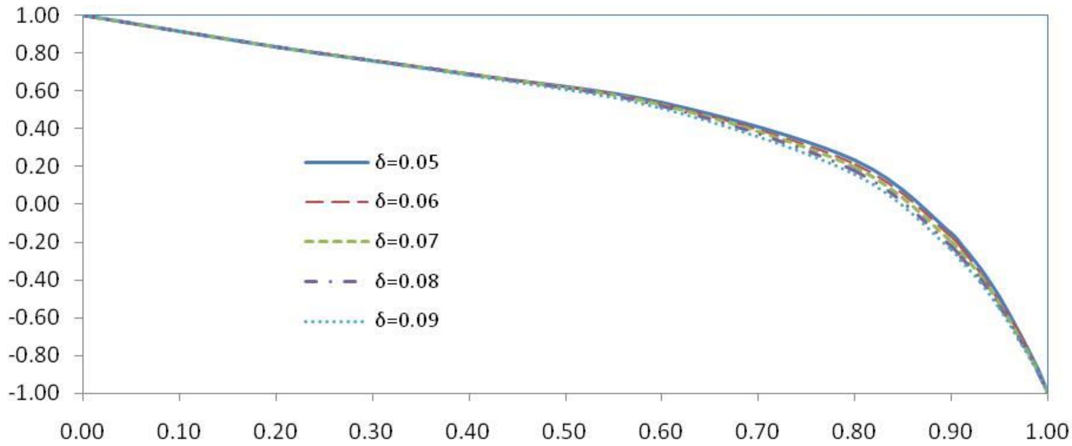

Figure 5. Graph for solution of example 3. for ε=0.1 and different values of δ

Table 5. Results for example 3. for (ε=0.1)

x(i)	δ=0.05	δ=0.06	δ=0.07	δ=0.08	δ=0.09
0.00	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.20	0.834777200	0.835099000	0.835302400	0.835389000	0.835362500
0.40	0.690885200	0.689934600	0.688679200	0.687143100	0.685356600
0.60	0.538671000	0.531944900	0.524874400	0.517544700	0.510033400
0.80	0.234541500	0.213745200	0.193839000	0.174820200	0.156673200
0.90	-0.151482900	-0.176135800	-0.198791300	-0.219670900	-0.238964600
0.91	-0.208951200	-0.233119900	-0.255241500	-0.275556900	-0.294269800
0.93	-0.338552700	-0.360849800	-0.381095900	-0.399557700	-0.416456400
0.95	-0.490753100	-0.509637000	-0.526647600	-0.542050100	-0.556060000
0.97	-0.669802000	-0.683231600	-0.695233100	-0.706023800	-0.715777500
0.99	-0.880742100	-0.886046400	-0.890749200	-0.894947900	-0.898719400
1.00	-1.000000000	-1.000000000	-1.000000000	-1.000000000	-1.000000000
Abs. Max. Err.	6.21E-02	6.55E-02	6.88E-02	7.21E-02	7.52E-02

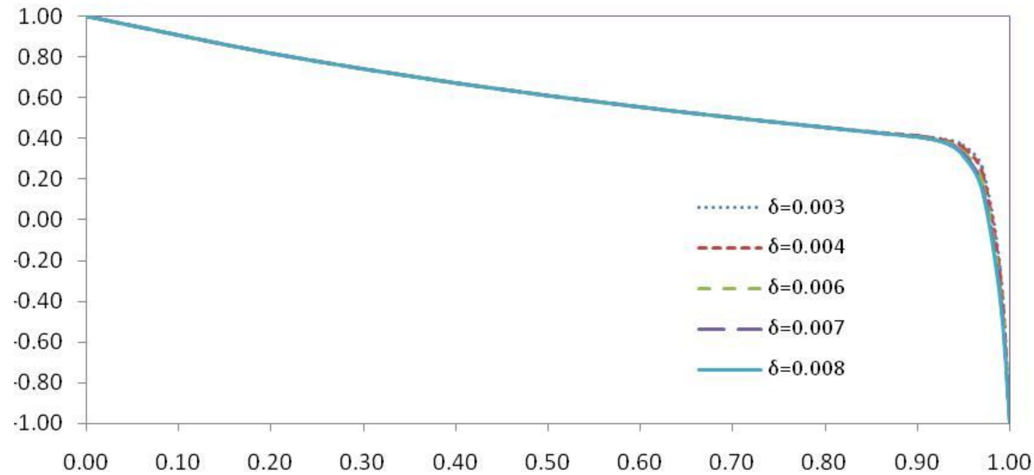

Figure 6. Graph for solution of example 3. for ε=0.01 and different values of δ

Table 6. Results for example 3. for (ε=0.01)

x(i)	δ=0.003	δ=0.004	δ=0.006	δ=0.007	δ=0.008
0.00	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.20	0.82088520	0.82102980	0.82131760	0.82146340	0.82160720
0.40	0.67385210	0.67408970	0.67456250	0.67480120	0.67503820
0.60	0.55315490	0.55344770	0.55403000	0.55432360	0.55461610
0.80	0.45407620	0.45439640	0.45503100	0.45534870	0.45566260
0.90	0.41089190	0.41084230	0.41020750	0.40956780	0.40867890
0.91	0.40623490	0.40583830	0.40419910	0.40289990	0.40125130
0.93	0.39392520	0.39168080	0.38533360	0.38123920	0.37655450
0.95	0.36504700	0.35697080	0.33792280	0.32720340	0.31584690
0.97	0.25527250	0.23250280	0.18643380	0.16355860	0.14098620
0.99	-0.24713310	-0.28218910	-0.34343290	-0.37028460	-0.39499700
1.00	-1.00000000	-1.00000000	-1.00000000	-1.00000000	-1.00000000
Abs. Max. Err.	6.33E-03	6.66E-03	7.69E-03	8.26E-03	8.81E-03

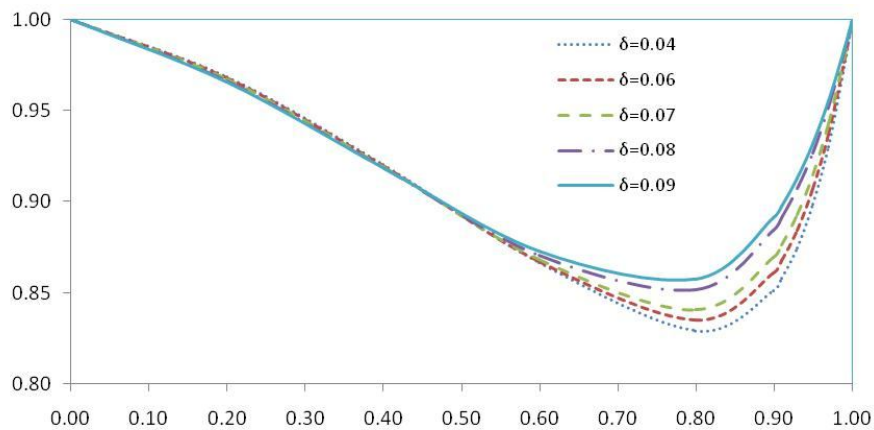

Figure 7. Graph for solution of example 4. for ε=0.1 and different values of δ

Table 7. Results for example 4. for (ε=0.1)

x(i)	δ=0.04	δ=0.06	δ=0.07	δ=0.08	δ=0.09
0.00	1.00000000	1.00000000	1.000000000	1.000000000	1.000000000
0.20	0.96844430	0.96776770	0.967159700	0.966189300	0.965820200
0.40	0.92049930	0.91977160	0.919231000	0.918754100	0.918788400
0.60	0.86676200	0.86721020	0.868063500	0.870810300	0.872573700
0.80	0.82921210	0.83479480	0.840524100	0.851925900	0.857417900
0.90	0.85130390	0.86088140	0.869557600	0.884560200	0.891029200
0.91	0.85783750	0.86758330	0.876304800	0.891178500	0.897520900
0.93	0.87483270	0.88452630	0.892991800	0.907033600	0.912888500
0.95	0.89834640	0.90720770	0.914758700	0.926939000	0.931904300
0.97	0.93032930	0.93714090	0.942803000	0.951683300	0.955221900
0.99	0.97334270	0.97625500	0.978615900	0.982215500	0.983617100
1.00	1.00000000	1.000000000	1.000000000	1.000000000	1.000000000
Abs. Ma. Err.	1.05E-03	8.43E-04	6.93E-04	4.75E-04	3.35E-04

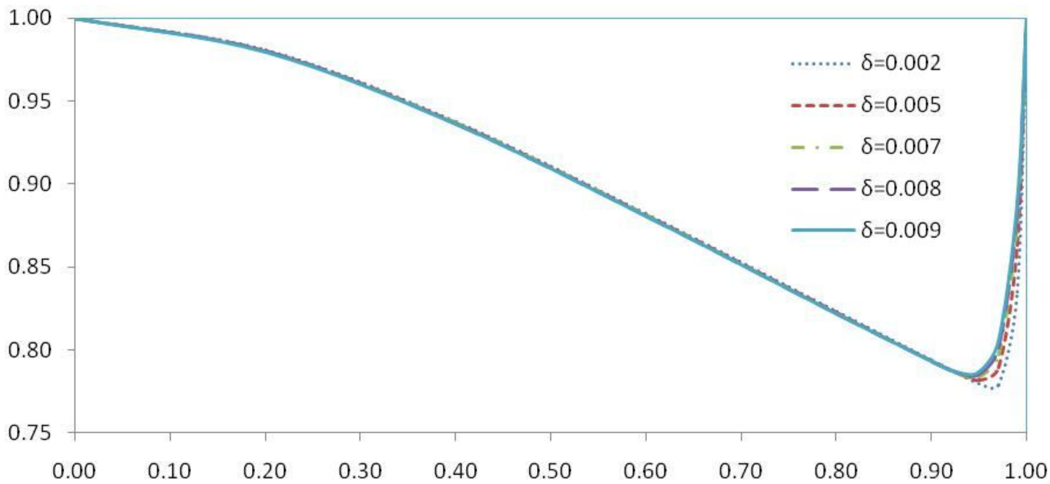

Figure 8. Graph for solution of example 4. for ε=0.01 and different values of δ

Table 8. Results for example 4. for (ε=0.01)

x(i)	δ=0.002	δ=0.005	δ=0.007	δ=0.008	δ=0.009
0.00	1.00000000	1.000000000	1.000000000	1.000000000	1.000000000
0.20	0.98077140	0.980351400	0.980072200	0.979933200	0.979793100
0.40	0.93757480	0.936943700	0.936524100	0.936316800	0.936106300
0.60	0.88229320	0.881605000	0.881148800	0.880922700	0.880693900
0.80	0.82317480	0.822526100	0.822097800	0.821885000	0.821671400
0.90	0.79387460	0.793298100	0.792992100	0.792882400	0.792809300
0.91	0.79098310	0.790454100	0.790246600	0.790214700	0.790240000
0.93	0.78525510	0.785076900	0.785428000	0.785780800	0.786255700
0.95	0.77994690	0.781588400	0.783898900	0.785376400	0.787035800
0.97	0.77930030	0.789205300	0.797152000	0.801266000	0.805398500
0.99	0.83506920	0.863399100	0.877574100	0.883631200	0.889124600
1.00	1.00000000	1.000000000	1.000000000	1.000000000	1.000000000
Abs. Max. Err.	1.05E-02	8.79E-03	7.52E-03	6.95E-03	6.42E-03

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