International Journal of Applied Science and Engineering
2013. 11, 4: 415-422

A Numerical Solution to the Nonlinear Fifth Order Boundary Value Problems

Ghazala Akram* and Hamood ur Rehman

Department of Mathematics, University of the Punjab, Lahore, Pakistan

Abstract: The reproducing kernel space method is used to solve the nonlinear fifth-order boundary-value problems. The approach provides the solution in the form of a convergent series with easily computable components. The present method compared with the others methods, reveals that the present method is more effective and convenient.

Keywords: Gram-Schmidt orthogonal process; reproducing kernel; nonlinear.

1. Introduction

Fifth order boundary value problem appears in the mathematical modelling of viscoelastic flows and other branches of mathematical, physical and engineering sciences [4, 5]. The conditions for the existence and uniqueness of the solution of such problems can be found in [6]. Gamel [2] analyzed Sinc-Galerkin method for the solution of fifth-order boundary value problems with two-point boundary conditions. Shen [3] solved fifth-order boundary value problems using the homotopy perturbation method. Siddiqi and Ghazala presented nonpolynomial spline method [7], polynomial sextic spline method [8] for the numerical solution of the fifth-order linear special case boundary value problems and the method is observed to be second-order convergent. Ghazala and Hamood [1] developed searching the least value (SLV) method for the solution of fifth order boundary value problem.

In this paper, a reproducing kernel method is used for the solution of nonlinear fifth order boundary value problem. The method discussed in this paper also applied to solve general fifth order boundary value problem. To the best of our knowledge, such boundary value problem involving $u^{(5)}(x), u^{(4)}(x), u^{(3)}(x), u^{(2)}(x)$ has not been already solved.

The following fifth order nonlinear boundary value problem (BVP) can be considered as

$$\left.\begin{array}{l} u^{(5)}(x)+\sum_{i=0}^{4} a_i(x)\, u^{(i)}(x)=f(x, u(x)), \\ u^{(i)}(0)=a_i,\ u^{(j)}(1)=\beta_j,\ i=0,1,2,\ j=0,1, \quad 0<x\le 1. \end{array}\right\} \tag{1}$$

where $a_i(x)$, $i = 0, 1, 2, \ldots, 4$ and $f(x,u(x))$ are continuous functions on [0, 1].

Let L be the differential operator and homogenization of the boundary conditions of system (1) can be transformed into the following form

* Corresponding author; e-mail: toghazala2003@yahoo.com

Received 31 December 2012
Revised 12 March 2013
Accepted 5 June 2013

$$\left.\begin{array}{l} Lu(x)=f(x,u(x)), \\ u^{(i)}(0)=u^{(i)}(1)=0, i=0,1, u^{(2)}(0)=0, \quad 0<x\le 1. \end{array}\right\} \tag{2}$$

Thus, the solution of system (2) provides the solution of the system (1).

The rest of this paper is organized as under:

In Section 2, the reproducing kernel spaces and the reproducing kernel function are given. The approximate solution of problem (2) is presented in Section 3. Three numerical examples are presented to demonstrate the accuracy of the method in Section 4.

2. Reproducing kernel spaces

i) The reproducing kernel space $W_2^6[0,1]$ is defined by $W_2^6[0,1]=\{u(x)/u^{(i)}(x), i=0,1,2...,5$ are absolutely continuous real valued functions in [0, 1], $u^{(6)}(x)\in L^2[0,1]\}$. The inner product and norm in $W_2^6[0,1]$ are given by

$$\langle u(x),v(x)\rangle=\sum_{i=0}^{5}u^{(i)}(0)v^{(i)}(0)+\int_0^1 u^{(6)}(x)v^{(6)}(x)dx \tag{3}$$

$$\|u(x)\|=\sqrt{\langle u(x),u(x)\rangle}, u(x), v(x)\in W_2^6[0,1] \tag{4}$$

2.1. Theorem

The space $W_2^6[0,1]$ is a reproducing kernel Hilbert space. That is $\forall u(y)\in W_2^6[0,1]$ and each fixed $x,y\in[0,1]$ there exists $R_x(y)\in W_2^6[0,1]$ s.t $\langle u(y),R_x(y)\rangle=u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W_2^6[0,1]$.

The reproducing kernel function $R_x(y)$ is given by

$$R_x(y)=\begin{cases} k(x,y)\sum_{i=0}^{11}c_i y^i, & y\le x, \\ k(y,x)\sum_{i=0}^{11}d_i y^i, & y>x. \end{cases} \tag{5}$$

Proof

Since $R_x(y)\in W_2^6[0,1]$ and from Eq. (3), it can be written as

$$\langle u(y),R_x(y)\rangle=\sum_{i=0}^{5}u^{(i)}(0)R_x^{(i)}(0)+\int_0^1 u^{(6)}(y)R_x^{(6)}(y)dy. \tag{6}$$

Since $R_x(y)\in W_2^6[0,1]$, it follows that

$$R_x^{(i)}(0)=R_x^{(i)}(1)=0, i=0,1, R_x^{(2)}(0)=0. \tag{7}$$

Also $u(x)\in W_2^6[0,1]$, gives $u^{(i)}(0)=u^{(i)}(1)=0, i=0,1, u^{(2)}(0)=0.$

If

$$\left.\begin{array}{l} R_x^{(m)}(1)=0, m=6,7,8,9, \\ R_x^{(3)}(0)-R_x^{(8)}(0)=0, R_x^{(4)}(0)+R_x^{(7)}(0)=0, R_x^{(5)}(0)-R_x^{(6)}(0)=0, \end{array}\right\} \tag{8}$$

then Eq. (6) implies that

$$\langle u(y), R_x(y)\rangle = \int_0^1 u(y) R_x^{(12)}(y)dy \tag{9}$$

For all $x \in [0,1]$, if $R_x(y)$ also satisfies

$$R_x^{(12)}(y) = \delta(y-x) \tag{10}$$

then

$$\langle u(y), R_x(y)\rangle = u(x). \tag{11}$$

When $y \neq x$, characteristic equation of Eq. (10) is given by λ^{12} then the characteristic values can be determined whose multiplicity is 12.

Let $R_x(y)$ satisfies

$$R_x^{(k)}(x+0) = R_x^{(k)}(x-0), k = 0,1,...,10. \tag{12}$$

Integrating (10) from $x-\varepsilon$ to $x+\varepsilon$ with respect to *y* and $x \to 0$, using jump degree of $R_x^{(12)}(y)$ at $y = x$,

$$R_x^{(12)}(x+0) - R_x^{(12)}(x-0) = 1. \tag{13}$$

The coefficients c_i and d_i (i=1, 2, …, 11) can be determined from Eqns. (7), (8), (12) and (13).

3. The exact and approximate solutions

In the problem (2), the linear operator $L: W_2^6[0,1] \to W_2^1[0,1]$ is bounded. Using the adjoint operator L* of L and choose a countable dense subset $T = \{x_1, x_2, ..., x_n, ...\} \subset [0,1]$ and let $\varphi_i(y) = Q_{x_i}(y)$, $i \in N$, then $\psi_i(x) = L^*\varphi_i(x)$, where $\psi_i(x) \in W_2^6[0,1]$.

3.1. Lemma

$\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^6[0,1]$ and $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof

For each fixed $u(x) \in W_2^6[0,1]$, let $\langle u(x), \psi_i(x)\rangle = 0, i = 1,2,...$ which implies $\langle u(x), (L^*\varphi_i)(x)\rangle = \langle (Lu)(x), Q_x(x)\rangle = (Lu)(x_i) = 0$

$$\langle u(x), L(x)\rangle = 0, i = 1,2,... \tag{14}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in [0, 1], (Lu)(x)=0, which implies u=0 from the existence of L^{-1}.

Using the reproducing property, it can be written as

$$\psi_i(x) = \langle \psi_i(y), R_x(y)\rangle = \langle \phi_i(y), LR_x(y)(x)\rangle = L_y R_x(y)|_{y=x_i}. \tag{15}$$

To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^6[0,1]$, Gram-Schmidt process can be used, as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), i = 1,2,3,... \tag{16}$$

3.2. Theorem 1

For all $u(x)\in W_2^6[0,1]$, the series $\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^6}$. On the other hand, if u(x) is the exact solution of the system (2), then $u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u(x_k))\overline{\psi}_i(x)$.

Proof

Since $u(x)\in W_2^6[0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system as

$$u(x)=\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x). \tag{17}$$

Since the space $W_2^6[0,1]$ is Hilbert space so series $\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^6}$.

From Eqns. (16) and (17), it can be written as

$$u(x)=\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\langle u(x),\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\rangle\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x),\varphi_k(x)\rangle\overline{\psi}_i(x).$$

If u(x) is the exact solution of Eq. (2) and $Lu=f(x,u(x))$, then

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u(x_k))\overline{\psi}_i(x).$$

The approximate solution obtained by the *n*-term intercept of the exact solution u(x), given by

$$u(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k,u(x_k))\overline{\psi}_i(x). \tag{18}$$

The problem (2) is nonlinear, then approximate solution of the problem (2) can be obtained using the following iteration formula:

$$\begin{cases}\text{Any fixed } u_0(x)\in W_2^6[0,1]\\ u_n(x)=\sum_{i=1}^{n}A_i\overline{\psi}_i(x),\end{cases} \tag{19}$$

where

$$\left.\begin{aligned}A_1&=\beta_{11}f(x_1,u_0(x_1)),\\ A_2&=\sum_{k=1}^{2}\beta_{2k}f(x_k,u_{k-1}(x_k)),\\ &\vdots\\ A_n&=\sum_{k=1}^{n}\beta_{nk}f(x_k,u_{k-1}(x_k)).\end{aligned}\right\} \tag{20}$$

3.3. Theorem 2

If i) $\|u(x)\|$ is bounded

ii) $\{x_i\}_{i=0}^{\infty}$ is dense in [0, 1]

iii) $f(x,u(x)) \in W_2^1[0,1]$ and $u(x) \in W_2^6[0,1]$ then $u_n(x)$ in Eq. (19) converges to the exact solution u(x) of the problem (2), where A_i are given by Eq. (20) and $u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$.

Proof

i) From Eq. (19), it can be written as $u_{n+1}(x) = u_n(x) + A_{n+1}\overline{\psi}_{n+1}(x)$ then the orthonormality of $\{\overline{\psi}(x)\}_{i=0}^{\infty}$ yields $\|u_{n+1}(x)\|^2 = \|u_n(x)\|^2 + \|A_{n+1}\|^2 = \sum_{i=1}^{n+1} A_i$.

From boundedness of $\|u_n(x)\|$ gives $\sum_{i=1}^{\infty} A_i < \infty$ i.e $\{A_i\} \in L^2, i = 1, 2, ...$

For $m > n, (u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp (u_{m+1} - u_n)$ leads to

$$\|u_m - u_n\|^2 = \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + ... + u_{n+1} - u_n\|^2$$
$$= \|u_m - u_{m-1}\|^2 + \|u_{m-1} - u_{m-2}\|^2 + ... + \|u_{n+1} - u_n\|^2$$
$$= \sum_{i=n+1}^{m} A_i \to 0, (n, m \to 0).$$

Considering the completeness of $W_2^6[0,1]$, there exists $u(x) \in W_2^6[0,1]$, such that $u_n(x) \to u(x), n \to \infty$.

ii) Using i) of **3.3. Theorem**, $u_n(x)$ converge uniformly to u(x). On taking limits in Eq. (19), it follows that $u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$.

It may be noted that

$$L_u(x_j) = \sum_{i=1}^{\infty} A_i \langle L\overline{\psi}_i(x), \phi_j(x) \rangle$$
$$= \sum_{i=1}^{\infty} A_i \langle \overline{\psi}_i(x), L^* \phi_j(x) \rangle$$
$$= \sum_{i=1}^{\infty} A_i \langle \overline{\psi}_i(x), \psi_j(x) \rangle.$$

Moreover,

$$\sum_{i=1}^{n} \beta_{nj} Lu(x_j) = \sum_{=1}^{\infty} A_i \left\langle \overline{\psi}_i(x), \sum_{j=1}^{n} \beta_{ij} \psi_j(x) \right\rangle$$
$$= \sum_{=1}^{\infty} A_i \langle \overline{\psi}_i(x), \overline{\psi}_n(x) \rangle = A_n.$$

If n=1, then $Lu(x_1) = f(x_1, u_0(x_1))$.

If n=2, then $\beta_{21}Lu(x_1)+\beta_{22}Lu(x_2)=\beta_{21}f(x_1,u_0(x_1))+\beta_{22}f(x_2,u_1(x_2))$.

It is clear that $Lu(x_2)=f(x_2,u_1(x_2))$.

Furthermore, it is easy to see by induction that

$$Lu(x_j)=f(x_j,u_{j-1}(x_j)) \tag{21}$$

Since, $\{x\}_{i=1}^{\infty}$ is dense on interval [0,1], for any $y\in[0,1]$, there exists subsequence such that $\{x_{nj}\}$, $x_{nj}\to y, y\to\infty$.

Let $y\to\infty$ in Eq. (21), and by the convergence of $u_n(x)$ and **2.3. Lemma**, gives

$$Lu(x)=f(x,u(x)) \tag{22}$$

that is u(x) is the solution of the problem (2) and

$$u_n(x)=\sum_{i=1}^{n}A_i\overline{\psi}_i(x), \tag{23}$$

where A_i are given by Eq. (20).

To illustrate the applicability and accuracy of our method, three numerical examples are constructed.

4. Numerical examples

Example 4.1.

The nonlinear fifth-order boundary value problem can be considered, as

$$\left.\begin{aligned}&u^{(5)}(x)=-24e^{-5u(x)}+\frac{48}{(1+x)^5},\\&u(0)=0,u^{(1)}(0)=1,u^{(2)}(0)=-1,\\&u(1)=\ln 2,u^{(1)}(1)=0.5.\end{aligned}\right\} \tag{24}$$

The exact solution of the Example 4.1. is u(x)=ln(1+x). The comparison of the errors in absolute values between the method developed in this paper and that of Gamel [2] is shown in Tables 1, 2.

Table 1. Absolute errors (|exact solution-approximate solution|) for problem (24)

x	Present method (n=10)	Gamel [2]
0.0	0	0
0806	1.38225E-07	0
0.2285	1.0026E-06	2.0E-05
0.3999	5.70443E-06	2.0E-05
0.5	8.21059E-06	4.0E-05
0.6395	9.0013E-06	1.0E-05
0.6923	8.21059E-06	2.0E-05
0.7714	6.19523E-06	3.0E-05
0.8836	2.33757E-06	2.0E-05
0.9447	6.32223E-07	5.0E-05
1.0	0	0

Table 2. Max. absolute error for problem (24)

Present method (*n*=10)	Gamel [2]
9.17818 E-06	5.0 E-05

Example 4.2.

The nonlinear fifth-order boundary value problem can be considered, as

$$u^{(5)}(x)+u^{(4)}(x)+e^{-2x}u^{2}(x)=2e^{x}+1,\quad 0<x<1,$$
$$u(0)=1, u(1)=e, u^{(1)}(0)=1, \tag{25}$$
$$u^{(1)}(1)=e, u^{(2)}(0)=1$$

The exact solution of the Example 4.2. is $u(x)=e^{x}$.

The comparison of the errors in absolute values between the method developed in this paper and that of Ghazala and Hamood [1], Gamel [2] and Shen [3] is shown in Table 3.

Table 3. Absolute errors (|exact solution-approximate solution|) for problem (25)

x	Present method (*n = 10*)	Ghazala and Hamood [1] (*n = 10*)	Gamel [2]	Shen [3]
0.0		0	0	0
0.01	1.20691 E -11	7.79 E -10	0	1.2 E -9
0.1184	1.59402E -08	7.83 E-7	0	7.0 E -6
0.1517	3.11814E -08	1.36 E-6	1.0 E-4	1.4 E -5
0.2410	1.01658E -07	3.005 E-6	0	4.6 E -4
0.3604	2.47487E -07	2.52E-6	1.0 E-4	1.0 E -4
0.4287	3.37393E -07	2.57 E-7	0	1.0 E -4
0.5	4.16695E -07	5.04 E-6	2.0 E-4	1.9 E -4
0.6395	4.68051E -07	1.58E-5	1.0 E-4	1.0 E -4
0.8482	2.03296E -07	1.33 E-5	2.0 E-4	9.8 E -5
0.9996	2.39419E -12	2.10 E-10	2.0 E-4	1.2 E -9
1.0		0	0	0

Example 4.3.

The nonlinear fifth-order boundary value problem can be considered, as

$$u^{(5)}(x)+xu^{(4)}(x)+(1+x)u^{(3)}(x)+u^{(2)}(x)+u^{(1)}(x)+e^{u(x)}u(x)=f(x),$$
$$u(0)=1, u^{(1)}(0)=0, u^{(2)}(0)=-2u(1)=0, u^{(1)}(1)=-e\cos 1. \tag{26}$$

The exact solution of the Example 4.3. is $u(x)=e_{x}(1-x)$ cosx and $f(x)=e_{x}((14+5x+4x^{2}+2x^{3})\cos x(3+11x+6x^{2}+2x^{3})\sin x$

The results are summarized in Table 4.

Table 4. Absolute errors (|exact solution-approximate solution|) for problem (26)

x	Exact solution	Approximate solution	$\lvert u(x)-u_{10}(x)\rvert$
0.0	1.0	1.0	0
0.16	0.97647	0.97647	3.347 E-7
0.32	0.902695	0.902699	3.717 E-6
0.48	0.776689	0.7767	1.159 E-5
0.64	0.601525	0.601544	1.922 E-5
0.8	0.387228	0.387246	1.797 E-5
0.96	0.152886	0.152891	5.797 E-6

5. Conclusions

In this paper, the reproducing kernel space method is developed for the solution of fifth order boundary value problem. The obtained results are compared with the solutions obtained from other methods and found that present method gives better results. The results revealed that the method is a powerful mathematical tool for the solution of fifth order boundary value problems. Numerical examples also show the accuracy of the method.

References

[1] Akram, G. and Rehman, H. U. 2011. Solution of fifth order boundary value problems in reproducing kernel space. *Middle-East Journal of Scientific Research*, 10, 2: 191-195.

[2] El-Gamel, M. 2007. Sinc and the numerical solution of fifth-order boundary value problems. *Applied Mathematics and Computation*, 187, 2: 1417-1433.

[3] Shen, S. 2007. Application of homotopy perturbation method to the fifth-order boundary value problems. *Int. J. Contemp. Math. Sciences*, 2, 25: 1227-1236.

[4] Davies, A. R., Karageoghis, A., and Phillips, T. N. 1988. Spectral Glarkien methods for the primary two-point boundary-value problems in modeling viscelastic flows. *International Journal for Numerical Methods in Engineering*, 26: 647-662.

[5] Karageoghis, A., Phillips, T. N., and Davies, A. R. 1998. spectral collocation methods for the primary two-point boundary-value problems in modeling viscelastic flows. *International Journal for Numerical Methods in Engineering*, 26: 805-813.

[6] Agarwal, R. P. 1986. "*Boundary value problems for high order differential equations*". World Scientific. Singapore.

[7] Siddiqi, S. S. and Akram, G. 2006. Solution of fifth order boundary value problems using nonpolynomial spline technique. *Applied Mathematics and Computation*, 175: 1574-1581.

[8] Siddiqi, S. S. and Akram, G. 2007. Sextic spline solutions of fifth order boundary value problems. *Applied Mathematics Letters*, 20, 5: 591-597.