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Sextic B-Spline Collocation Method for Eighth Order Boundary Value Problems

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Abstract: A finite element method involving collocation method with sextic B-splines as basis functions has been developed to solve eighth order boundary value problems. The sixth order, seventh order and eighth order derivatives for the dependent variable are approximated by the central differences of fifth order derivatives. The basis functions are redefined into a new set of basis functions which in number match with the number of collocated points selected in the space variable domain. The proposed method is tested on several linear and non-linear boundary value problems. The solution of a non-linear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique. Numerical results obtained by the present method are in good agreement with the exact solutions available in the literature.

Keywords: Collocation method; sextic B-spline; basis function; eighth order boundary value problem; absolute error.

1. Introduction

Generally, the eighth-order boundary value problems are known to arise in the Mathematics, Physics and Engineering Sciences [1, 2]. In the book written by Chandrasekhar [3], we can find that when an infinite horizontal layer of fluid is heated from below and under the action of rotation, instability sets in. When this instability sets as an ordinary convection, the ordinary differential equation is a sixth-order ordinary differential equation. When this instability sets as an over stability, it is modelled by an eighth-order ordinary differential equation. An eighth-order differential equation occurring in torsion vibration of uniform beams was investigated by [4].

In this paper, we developed a collocation method with sextic B-splines as basis functions for getting the numerical solution of a general linear eighth order boundary value problem

$$a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x)+a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+a_5(x)y'''(x)$$
$$+a_6(x)y''(x)+a_7(x)y'(x)+a_8(x)y(x)=b(x),\ c<x<d \tag{1a}$$

subject to the boundary conditions

$$y(c)=A_0,\ y(d)=C_0,\ y'(c)=A_1,\ y'(d)=C_1,$$
$$y''(c)=A_2,\ y''(d)=C_2,\ y'''(c)=A_3,\ y'''(d)=C_3, \tag{1b}$$

where A_0, C_0, A_1, C_1, A_2, C_2, A_3, C_3 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$, $a_7(x)$, $a_8(x)$ and $b(x)$ are all continuous functions defined on the interval [c, d].

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The existence and uniqueness of solution of such type of boundary value problems can be found in the book written by Agarwal [5]. Over the years, there are several authors who worked on these types of boundary value problems by using different methods. For example, Boutayeb and Twizell [6] developed finite difference methods for the solution of eighth-order boundary value problems. Twizell et. al. [7] developed numerical methods for eighth, tenth and twelfth order eigenvalue problems arising in thermal instability. Siddiqi and Twizell [8] presented the solution of eighth order boundary value problem using octic spline. Inc and Evans [9] presented the solutions of eighth order boundary value problems using adomian decomposition method. Siddiqi and Ghazala Akram [10, 11] presented the solutions of eighth-order linear special case boundary value problems using nonic spline and nonpolynomial nonic spline respectively.

Further, Scott and Watts [12] developed a numerical method for the solution of linear boundary value problems using a combination of superposition and orthonormalization. Scott and Watts [13] described several computer codes that were developed using the superposition and orthonormalization technique and invariant imbedding. Watson and Scott [14] solved non-linear two point boundary value problems for spline collocation approximation by homotopy method. Liu and Wu [15] presented differential quadrature solutions of eighth-order differential equations. He [16-20] developed the variational iteration technique for solving non linear initial and boundary value problems. Wazwaz [21] have used the modified decomposition method for approximating solution of higher-order boundary value problems with two point boundary conditions. Modified adomian decomposition method was used in [22] to find the analytical solution of linear and nonlinear boundary value problems of eighth order.

The above studies are concerned to solve special case eighth order boundary value problems by using octic or higher order B-splines. In this paper, sextic B-splines as basis functions have been used to solve a general eighth order boundary value problem of the type (1).

In section 2 of this paper, the justification for using the collocation method has been mentioned. In section 3, the definition of sextic B-splines has been described. In section 4, description of the collocation method with sextic B-splines as basis functions has been presented and in section 5, solution procedure to find the nodal parameters is presented. In section 6, numerical examples of both linear and non-linear boundary value problems are presented. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solutions of linear boundary value problems generated by quasilinearization technique [23]. Finally, the last section is dealt with conclusions of the paper.

2. Justification for using collocation method

In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods such as Ritz's approach, Galerkin's approach, least squares method and collocation method etc. The collocation method seeks an approximate solution by requiring the residual of the differential equation to be identically zero at N selected points in the given space variable domain where N is the number of basis functions in the basis [24]. That means, to get an accurate solution by the collocation method, one needs a set of basis functions which in number match with the number of collocation points selected in the given space variable domain. Further, the collocation method is the easiest to implement among the variational methods of FEM. Hence this motivated us to solve a eighth order boundary value problem of type (1) by collocation method with sextic B-splines as basis functions.

3. Definition of sextic B-splines

The cubic B-splines are defined in [26, 27]. The existence of the cubic spline interpolate $s(x)$ to a function f(x) in a closed interval [c, d] for spaced knots (need not be evenly spaced) $c=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=d$ is established by constructing it. The construction of $s(x)$ is done with the help of the cubic B-splines. Introduce six additional knots $x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}$ and x_{n+3} such that $x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}$.

Now the cubic B-splines $B_i(x)'s$ are defined by

$$B_i(x)=\begin{cases}\displaystyle\sum_{r=i-2}^{i+2}\frac{(x_r-x)^3{}_+}{\pi'(x_r)} & for\ x\in[x_{i-2},x_{i+2}]\\ 0 & otherwise\end{cases}$$

where $(x_r-x)^3{}_+=\begin{cases}(x_r-x)^3 & for\quad x_r\ge x\\ 0 & for\quad x_r\le x\end{cases}$

and $\pi(x)=\displaystyle\prod_{r=i-2}^{i+2}(x\text{-}x_r)$.

It can be shown that the set $\{B_{-1}(x), B_0(x),\ldots, B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines[25]. The cubic B-splines are the unique non-zero splines of smallest compact support with knots at $x_{-3}<x_{-2}<x_{-1}<x_0<\ldots<x_n<x_{n+1}<x_{n+2}<x_{n+3}$.

Here s(x) is a two times continuously differentiable polynomial spline of degree 3 satisfying the properties $s'(x_0)=f'(x_0)$, $s(x_i)=f(x_i)$, i=0, 1, 2, …, n, $s'(x_n) = f'(x_n)$.

In a similar analogue, the existence of the sextic spline interpolate $s(x)$ to a function f(x) in a closed interval [c, d] for spaced knots (need not be evenly spaced) $c=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=d$ is established by constructing it. The construction of $s(x)$ is done with the help of the sextic B-splines. Introduce twelve additional knots $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ and x_{n+6} such that $x_{-6}<x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}<x_{n+6}$.

Now the sextic B-splines $B_i(x)'s$ are defined by

$$B_i(x)=\begin{cases}\displaystyle\sum_{r=i-3}^{i+4}\frac{(x_r-x)^6{}_+}{\pi'(x_r)} & for\ x\in[x_{i-3},x_{i+4}]\\ 0 & otherwise\end{cases}$$

where $(x_r-x)^6{}_+=\begin{cases}(x_r-x)^6 & for\quad x_r\ge x\\ 0 & for\quad x_r\le x\end{cases}$

and $\pi(x)=\displaystyle\prod_{r=i-3}^{i+4}(x\text{-}x_r)$

It can be shown that the set $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines[25]. The sextic B-splines are the unique non-zero splines of smallest compact support with knots at $x_{-6}< x_{-5}< x_{-4}< x_{-3}< x_{-2}< x_{-1}< x_0 <\ldots<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}< x_{n+6}$.

Here s(x) is a five times continuously differentiable polynomial spline of degree 5 satisfying the properties $s^j(x_0)=f^j(x_0)$, j=1, 2, 3, $s(x_i)=f(x_i)$, i=0, 1, 2, …, n, $s^j(x_n)=f^j(x_n)$, j=1, 2.

4. Description of the method

To solve the boundary value problem (1) by the collocation method with sextic B-splines as basis functions, we define the approximation for *y(x)* as

$$y(x) = \sum_{j=-3}^{n+2} \alpha_j B_j(x) \tag{2}$$

where α_j*'s* are nodal parameters to be determined. In the present method, the internal mesh points $x_1, x_2, \ldots, x_{n-2}$ are selected as the collocation points. In collocation method, the number of basis functions in the approximation should match with the number of selected collocation points [24]. Here the number of basis functions in the approximation (2) is $n+6$, where as the number of selected collocation points is $n-2$. So, there is a need to redefine the basis functions into a new set of basis functions which in number match with the number of selected collocation points. The procedure for redefining the basis functions is as follows:

Using the sextic B-splines described in section 3 and the Dirichlet boundary conditions of (1), we get the approximate solution at the boundary points as

$$y(c) = y(x_0) = \sum_{j=-3}^{2} \alpha_j B_j(x_0) = A_0 \tag{3}$$

$$y(d) = y(x_n) = \sum_{j=n-3}^{n+2} \alpha_j B_j(x_n) = C_0 \tag{4}$$

Eliminating α_{-3} and α_{n+2} from the equations (2), (3) and (4), we get the approximation for *y(x)* as

$$y(x) = w_1(x) + \sum_{j=-2}^{n+1} \alpha_j P_j(x) \tag{5}$$

where $w_1(x) = \dfrac{A_0}{B_{-3}(x_0)} B_{-3}(x) + \dfrac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$

and $P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-3}(x_0)} B_{-3}(x), & for \quad j = -2, -1, 0, 1, 2 \\ B_j(x), & for \quad j = 3, 4, \ldots, n-4 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & for \quad j = n-3, n-2, n-1, n, n+1. \end{cases}$

Using the Neumann boundary conditions of (1) to the approximation *y(x)* in (5), we get

$$y'(c)=y'(x_0)=w_1'(x_0)+\alpha_{-2}P_{-2}'(x_0)+\alpha_{-1}P_{-1}'(x_0)+\alpha_0 P_0'(x_0)+\alpha_1 P_1'(x_0)+\alpha_2 P_2'(x_0)=A_1 \tag{6}$$

$$y'(d)=y'(x_n)=w_1'(x_n)+\alpha_{n-3}P_{n-3}'(x_n)+\alpha_{n-2}P_{n-2}'(x_n)+\alpha_{n-1}P_{n-1}'(x_n)+\alpha_n P_n'(x_n) +\alpha_{n+1}P_{n+1}'(x_n)=C_1. \tag{7}$$

Now, eliminating α_{-2} and α_{n+1} from the equations (5), (6) and (7), we get the approximation for *y(x)* as

$$y(x) = w_2(x) + \sum_{j=-1}^{n} \alpha_j Q_j(x) \tag{8}$$

where $w_2(x) = w_1(x) + \frac{A_1 - w_1'(x_0)}{P_{-2}'(x_0)} P_{-2}(x) + \frac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$

and $Q_j(x) = \begin{cases} P_j(x) - \frac{P_j'(x_0)}{P_{-2}'(x_0)} P_{-2}(x), & \text{for } j=-1,0,1,2 \\ P_j(x), & \text{for } j=3,4,\dots,n\text{-}4 \\ P_j(x) - \frac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & \text{for } j=n-3, n\text{-}2, n\text{-}1, n. \end{cases}$

Using the boundary conditions $y''(c)=A_2$ and $y''(d)=C_2$ of (1) to the approximate solution $y(x)$ in (8), we get

$$y''(c)=y''(x_0)=w_2''(x_0)+\alpha_{-1}Q_{-1}''(x_0)+\alpha_0Q_0''(x_0)+\alpha_1Q_1''(x_0)+\alpha_2Q_2''(x_0)=A_2 \tag{9}$$

$$y''(d)=y''(x_n)=w_2''(x_n)+\alpha_{n-3}Q_{n-3}''(x_n)+\alpha_{n-2}Q_{n-2}''(x_n)+\alpha_{n-1}Q_{n-1}''(x_n)+\alpha_nQ_n''(x_n)=C_2. \tag{10}$$

Now, eliminating α_{-1} and α_n from the equations (8), (9) and (10), we get the approximation for $y(x)$ as

$$y(x) = w_3(x) + \sum_{j=0}^{n-1} \alpha_j R_j(x) \tag{11}$$

where $w_3(x) = w_2(x) + \frac{A_2 - w_2''(x_0)}{Q_{-1}''(x_0)} Q_{-1}(x) + \frac{C_2 - w_2''(x_n)}{Q_n''(x_n)} Q_n(x)$

and $R_j(x) = \begin{cases} Q_j(x) - \frac{Q_j''(x_0)}{Q_{-1}''(x_0)} Q_{-1}(x), & \text{for } j=0, 1, 2 \\ Q_j(x), & \text{for } j=3,4,\dots,n\text{-}4 \\ Q_j(x) - \frac{Q_j''(x_n)}{Q_n''(x_n)} Q_n(x), & \text{for } j=n-3, n\text{-}2, n\text{-}1. \end{cases}$

Using the boundary conditions $y'''(c)=A_3$ and $y'''(d)=C_3$ to the approximate solution $y(x)$ in (11), we get

$$y'''(c)=y'''(x_0)=w_3'''(x_0)+\alpha_0R_0'''(x_0)+\alpha_1R_1'''(x_0)+\alpha_2R_2'''(x_0)=A_3 \tag{12}$$

$$y'''(d)=y'''(x_n)=w_3'''(x_n)+\alpha_{n-3}R_{n-3}'''(x_n)+\alpha_{n-2}R_{n-2}'''(x_n)+\alpha_{n-1}R_{n-1}'''(x_n)=C_3. \tag{13}$$

Now, eliminating α_0 and α_{n-1} from the equations (11), (12) and (13), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=1}^{n-2} \alpha_j \widetilde{B}_j(x) \tag{14}$$

where $w(x) = w_3(x) + \dfrac{A_3 - w_3'''(x_0)}{R_0'''(x_0)} R_0(x) + \dfrac{C_3 - w_3'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x)$

and $\widetilde{B}_j(x) = \begin{cases} R_j(x) - \dfrac{R_j'''(x_0)}{R_0'''(x_0)} R_0(x), & for\ \ j=1,\ 2 \\ R_j(x), & for\ \ j=3,4,\ldots,n\text{-}4 \\ R_j(x) - \dfrac{R_j'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x), & for\ \ j=n-3,\ n\text{-}2. \end{cases}$

Now, the new basis functions for the approximation $y(x)$ are $\{\widetilde{B}_j(x), j = 1, 2, \ldots, n-2\}$ and they are in number match with the number of selected collocated points. Since the approximation for $y(x)$ in (14) is a sextic approximation, let us approximate $y^{(6)}$, $y^{(7)}$ and $y^{(8)}$ at the selected collocation points with finite differences as

$$y_i^{(8)} = \frac{y_{i+3}^{(5)} - 3y_{i+2}^{(5)} + 3y_{i+1}^{(5)} - y_i^{(5)}}{h^3} \quad \text{for } i=1 \tag{15}$$

$$y_i^{(8)} = \frac{y_{i+2}^{(5)} - 2y_{i+1}^{(5)} + 2y_{i-1}^{(5)} - y_{i-2}^{(5)}}{2h^3} \quad \text{for } i=2, 3, \ldots, \text{n-2} \tag{16}$$

$$y_i^{(7)} = \frac{y_{i+1}^{(5)} - 2y_i^{(5)} + y_{i-1}^{(5)}}{h^2} \quad \text{for } i=1, 2, \ldots, \text{n-2} \tag{17}$$

and $$y_i^{(6)} = \frac{y_{i+1}^{(5)} - y_{i-1}^{(5)}}{2h} \quad \text{for } i=1, 2, \ldots, \text{n-2} \tag{18}$$

where $$y_i = y(x_i) = w(x_i) + \sum_{j=1}^{n-2} \alpha_j \widetilde{B}_j(x_i). \tag{19}$$

Now applying collocation method to (1), we get

$$a_0(x_i)y_i^{(8)} + a_1(x_i)y_i^{(7)} + a_2(x_i)y_i^{(6)} + a_3(x_i)y_i^{(5)} + a_4(x_i)y_i^{(4)}$$
$$+ a_5(x_i)y_i''' + a_6(x_i)y_i'' + a_7(x_i)y_i' + a_8(x_i)y_i = b(x_i)$$

for i=1, 2, …, n-2. (20)

Using (15) and (17) to (19) in (20), we get

$$\frac{a_0(x_i)}{h^3}\left[\left\{w^{(5)}(x_{i+3}) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_{i+3})\right\} - 3\left\{w^{(5)}(x_{i+2}) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_{i+2})\right\}\right.$$
$$\left. + 3\left\{w^{(5)}(x_{i+1}) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_{i+1})\right\} - \left\{w^{(5)}(x_i) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_i)\right\}\right]$$
$$+ \frac{a_1(x_i)}{h^2}\left[\left\{w^{(5)}(x_{i+1}) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_{i+1})\right\} - 2\left\{w^{(5)}(x_i) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_i)\right\} + \left\{w^{(5)}(x_{i-1}) + \sum_{j=1}^{n-2}\alpha_j \widetilde{B}_j^{(5)}(x_{i-1})\right\}\right]$$

$$+\frac{a_2(x_i)}{2h}\left[\left\{w^{(5)}(x_{i+1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+1})\right\}-\left\{w^{(5)}(x_{i-1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-1})\right\}\right]$$

$$+a_3(x_i)\left[w^{(5)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_i)\right]$$

$$+\frac{a_2(x_i)}{2h}\left[\left\{w^{(5)}(x_{i+1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+1})\right\}-\left\{w^{(5)}(x_{i-1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-1})\right\}\right]$$

$$+a_3(x_i)\left[w^{(5)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_i)\right]+a_4(x_i)\left[w^{(4)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(4)}(x_i)\right]$$

$$+a_5(x_i)\left[w'''(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j'''(x_i)\right]+a_6(x_i)\left[w''(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j''(x_i)\right]$$

$$+a_7(x_i)\left[w'(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j'(x_i)\right]+a_8(x_i)\left[w(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j(x_i)\right]$$

$$=b(x_i)\qquad \text{for } i=1. \tag{21}$$

Now using (16) to (19) in (20), we get

$$\frac{a_0(x_i)}{2h^3}\left[\left\{w^{(5)}(x_{i+2})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+2})\right\}-2\left\{w^{(5)}(x_{i+1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+1})\right\}\right.$$

$$\left.+2\left\{w^{(5)}(x_{i-1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-1})\right\}-\left\{w^{(5)}(x_{i-2})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-2})\right\}\right]$$

$$+\frac{a_1(x_i)}{h^2}\left[\left\{w^{(5)}(x_{i+1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+1})\right\}-2\left\{w^{(5)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_i)\right\}\right.$$

$$\left.+\left\{w^{(5)}(x_{i-1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-1})\right\}\right]$$

$$+\frac{a_2(x_i)}{2h}\left[\left\{w^{(5)}(x_{i+1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i+1})\right\}-\left\{w^{(5)}(x_{i-1})+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_{i-1})\right\}\right]$$

$$+a_3(x_i)\left[w^{(5)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(5)}(x_i)\right]+a_4(x_i)\left[w^{(4)}(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j^{(4)}(x_i)\right]$$

$$+a_5(x_i)\left[w'''(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j'''(x_i)\right]+a_6(x_i)\left[w''(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j''(x_i)\right]$$

$$+a_7(x_i)\left[w'(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j'(x_i)\right]+a_8(x_i)\left[w(x_i)+\sum_{j=1}^{n-2}\alpha_j\widetilde{B}_j(x_i)\right]$$

$$=b(x_i)\qquad \text{for } i=2,\ 3,\ \ldots,\ n-2. \tag{22}$$

Rearranging the terms and writing the system of equations (21) and (22) in the matrix form, we get

$A\alpha=B$ (23)

where $A=[a_{ij}]$; (24)

$$a_{ij}=\widetilde{B}_j^{(5)}(x_{i-1})\left(\frac{a_1(x_i)}{h^2}-\frac{a_2(x_i)}{2h}\right)+\widetilde{B}_j^{(5)}(x_i)\left(-\frac{a_0(x_i)}{h^3}-2\frac{a_1(x_i)}{h^2}+a_3(x_i)\right)$$
$$+\widetilde{B}_j^{(5)}(x_{i+1})\left(3\frac{a_0(x_i)}{h^3}+\frac{a_1(x_i)}{h^2}+\frac{a_2(x_i)}{2h}\right)+\widetilde{B}_j^{(5)}(x_{i+2})\left(-3\frac{a_0(x_i)}{h^3}\right)$$
$$+\widetilde{B}_j^{(5)}(x_{i+3})\left(\frac{a_0(x_i)}{h^3}\right)+\widetilde{B}_j^{(4)}(x_i)a_4(x_i)+\widetilde{B}_j'''(x_i)a_5(x_i)+\widetilde{B}_j''(x_i)a_6(x_i)$$
$$+\widetilde{B}_j'(x_i)a_7(x_i)+\widetilde{B}_j(x_i)a_8(x_i) \quad for\, i=1,\ j=1,\ 2,\ \ldots,n\text{-}2$$

$$a_{ij}=\widetilde{B}_j^{(5)}(x_{i-2})\left(-\frac{a_0(x_i)}{2h^3}\right)+\widetilde{B}_j^{(5)}(x_{i-1})\left(\frac{a_0(x_i)}{h^3}+\frac{a_1(x_i)}{h^2}-\frac{a_2(x_i)}{2h}\right)$$
$$+\widetilde{B}_j^{(5)}(x_i)\left(-2\frac{a_1(x_i)}{h^2}+a_3(x_i)\right)+\widetilde{B}_j^{(5)}(x_{i+1})\left(-2\frac{a_0(x_i)}{2h^3}+\frac{a_1(x_i)}{h^2}+\frac{a_2(x_i)}{2h}\right)$$
$$+\widetilde{B}_j^{(5)}(x_{i+2})\left(\frac{a_0(x_i)}{2h^3}\right)+\widetilde{B}_j^{(4)}(x_i)a_4(x_i)+\widetilde{B}_j'(x_i)a_7(x_i)+\widetilde{B}_j(x_i)a_8(x_i)$$

for i = 2, 3, …, n - 2, j = 1, 2, …, n - 2, B = [bi]; (25)

$$b_i=b(x_i)-[w^{(5)}(x_{i-1})\left(\frac{a_1(x_i)}{h^2}-\frac{a_2(x_i)}{2h}\right)+w^{(5)}(x_i)\left(-\frac{a_0(x_i)}{h^3}-2\frac{a_1(x_i)}{h^2}+a_3(x_i)\right)$$
$$+w^{(5)}(x_{i+1})\left(3\frac{a_0(x_i)}{h^3}+\frac{a_1(x_i)}{h^2}+\frac{a_2(x_i)}{2h}\right)+w^{(5)}(x_{i+2})\left(-3\frac{a_0(x_i)}{h^3}\right)$$
$$+w^{(5)}(x_{i+3})\left(\frac{a_0(x_i)}{h^3}\right)+w^{(4)}(x_i)a_4(x_i)+w'''(x_i)a_5(x_i)+w''(x_i)a_6(x_i)$$
$$+w'(x_i)a_7(x_i)+w(x_i)a_8(x_i)\,]\text{ for } i=1$$

$$b_i=b(x_i)-[w^{(5)}(x_{i-2})\left(-\frac{a_0(x_i)}{2h^3}\right)+w^{(5)}(x_{i-1})\left(\frac{a_0(x_i)}{h^3}+\frac{a_1(x_i)}{h^2}-\frac{a_2(x_i)}{2h}\right)$$
$$+w^{(5)}(x_i)\left(-2\frac{a_1(x_i)}{h^2}+a_3(x_i)\right)+w^{(5)}(x_{i+1})\left(-2\frac{a_0(x_i)}{2h^3}+\frac{a_1(x_i)}{h^2}+\frac{a_2(x_i)}{2h}\right)$$
$$+w^{(5)}(x_{i+2})\left(\frac{a_0(x_i)}{2h^3}\right)+w^{(4)}(x_i)a_4(x_i)+w'''(x_i)a_5(x_i)+w''(x_i)a_6(x_i)$$
$$+w'(x_i)a_7(x_i)+w(x_i)a_8(x_i)\,]$$

for $i=2,\ 3,\ \ldots,\ n\text{-}2$ and $\alpha=[\alpha_1,\ \alpha_2,\ldots,\ \alpha_{n-2}]^T$.

5. Solution procedure to find the nodal parameters

The basis function $\widetilde{B}_i(x)$ is defined only in the interval [x_{i-3}, x_{i+4}] and outside of this interval it is zero. Also at the end points of the interval [x_{i-3}, x_{i+4}] the basis function $\widetilde{B}_i(x)$ vanishes. Therefore, $\widetilde{B}_i(x)$ is having non-vanishing values at the mesh points $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}$ and zero at the other mesh points. The first five derivatives of $\widetilde{B}_i(x)$ also have the same nature at the mesh points as in the case of $\widetilde{B}_i(x)$. In view of this, $\widetilde{B}_j(x_{i-2})$ and its first five derivatives are having non-vanishing values for $j=i-5, i-4, i-3, i-2, i-1, i$. Similarly, $\widetilde{B}_j(x_{i-1})$ and its first five derivatives are having non-vanishing values for $j=i-4, i-3, i-2, i-1, i, i+1$. $\widetilde{B}_j(x_i)$ and its first five derivatives are having non-vanishing values for $j=i-3, i-2, i-1, i, i+1, i+2$. $\widetilde{B}_j(x_{i+1})$ and its first five derivatives are having non-vanishing values for $j=i-2, i-1, i, i+1, i+2, i+3$. $\widetilde{B}_j(x_{i+2})$ and its first five derivatives are having non-vanishing values for $j=i-1, i, i+1, i+2, i+3, i+4$. $\widetilde{B}_j(x_{i+3})$ and its first five derivatives are having non-vanishing values for $j=i, i+1, i+2, i+3, i+4, i+5$. Using these facts, we can say that the matrix A defined in (24) is an eleven diagonal band matrix. Therefore, the system of equations (23) is an eleven diagonal band system in $\alpha_i's$. The nodal parameters α_i's can be obtained by using band matrix solution package. We have used the FORTRAN-90 programming to solve the boundary value problem (1) by the proposed method.

6. Numerical examples

To demonstrate the applicability of the proposed method for solving the eighth order boundary value problems of type (1), we considered six examples of which four are linear and two are non linear boundary value problems. Out of the four linear problems, two problems are of special case boundary value boundary problems with constant coefficients, one problem is of special case boundary value boundary problem with variable coefficient and one problem is of most general boundary value problem with constant coefficients. Numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1.

Consider the linear boundary value problem

$$y^{(8)}-y=-8e^x,\ 0<x<1 \tag{26}$$

subject to the boundary conditions

$$y(0)=1,\ y(1)=0,\ y'(0)=0,\ y'(1)=-e,\ y''(0)=-1,\ y''(1)=-2e,\ y'''(0)=-2,\ y'''(1)=-3e. \tag{27}$$

The above problem is a special case eighth order boundary value problem with constant coefficients. The exact solution for the above problem is given by $y(x)=(1-x)e^x$. The proposed method is tested on this problem where the domain [0,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 1. The maximum absolute error obtained by the proposed method is 2.110004×10^{-5}.

Table 1. Numerical results for example 1

x	Exact Solution	Absolute error by the proposed method
0.1	9.946538E-01	1.430511E-06
0.2	9.771222E-01	4.291534E-06
0.3	9.449012E-01	1.788139E-05
0.4	8.950948E-01	2.110004E-05
0.5	8.243606E-01	1.651049E-05
0.6	7.288475E-01	1.841784E-05
0.7	6.041259E-01	1.257658E-05
0.8	4.451082E-01	8.106232E-06
0.9	2.459602E-01	7.957220E-06

Example 2.

Consider the linear boundary value problem

$$y^{(8)}+xy=-(48+15x+x^3)e^x,\ 0<x<1 \tag{28}$$

subject to the boundary conditions

$$\begin{aligned} &y(0)=0, \quad y(1)=0,\\ &y'(0)=1, \quad y'(1)=-e,\\ &y''(0)=0, \quad y''(1)=-4e,\\ &y'''(0)=-3, \quad y'''(1)=-9e. \end{aligned} \tag{29}$$

The above problem is a special case eighth order boundary value problem with variable coefficients. The exact solution for the above problem is given by $y(x)=x(1-x)e^x$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 2. The maximum absolute error obtained by the proposed method is 9.343028×10^{-6}.

Table 2. Numerical results for example 2

x	Exact Solution	Absolute error by the proposed method
0.1	9.946539E-02	2.980232E-07
0.2	1.954244E-01	2.682209E-07
0.3	2.834704E-01	4.202127E-06
0.4	3.580379E-01	4.619360E-06
0.5	4.121803E-01	2.473593E-06
0.6	4.373085E-01	8.136034E-06
0.7	4.228881E-01	7.987022E-06
0.8	3.560865E-01	7.748604E-06
0.9	2.213642E-01	9.343028E-06

Example 3.

Consider the linear boundary value problem

$$y^{(8)}+y^{(7)}+2y^{(6)}+2y^{(5)}+2y^{(4)}+2y'''+2y''+y'+y = 14\cos x-16\sin x-4x\sin x,\ 0<x<1 \quad (30)$$

subject to the boundary conditions

$$\begin{aligned} &y(0)=0, && y(1)=0, \\ &y'(0)=-1, && y'(1)=2\sin 1, \\ &y''(0)=0, && y''(1)=4\cos 1+2\sin 1, \\ &y'''(0)=7, && y'''(1)=6\cos 1-6\sin 1. \end{aligned} \quad (31)$$

The above problem is an example for most general eighth order boundary value problem with constant coefficients. The exact solution for the above problem is given by $y(x)=(x^2-1)\sin x$. The proposed method is tested on this problem where the domain [0,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 3. The maximum absolute error obtained by the proposed method is 4.768372×10^{-6}.

Table 3. Numerical results for example 3

x	Exact Solution	Absolute error by the proposed method
0.1	-9.883508E-02	2.458692E-07
0.2	-1.907226E-01	4.917383E-07
0.3	-2.689234E-01	2.682209E-06
0.4	-3.271114E-01	1.996756E-06
0.5	-3.595692E-01	6.556511E-07
0.6	-3.613712E-01	2.861023E-06
0.7	-3.285510E-01	2.980232E-06
0.8	-2.582482E-01	3.397465E-06
0.9	-1.488321E-01	4.768372E-06

Example 4.

Consider the linear boundary value problem

$$y^{(8)}-16y=-4,\ -1<x<1 \quad (32)$$

subject to the boundary conditions

$$\begin{aligned} &y(-1)=0,\ y(1)=0, \\ &y'(-1)=(\sinh 2-\sin 2)/4(\cosh 2+\cos 2),\ y'(1)=-(\sinh 2-\sin 2)/4(\cosh 2+\cos 2), \\ &y''(-1)=0,\ y''(1)=0, \\ &y'''(-1)=-(\cos 1\sin 1+\cosh 1\sinh 1)/(\cosh 2+\cos 2), \\ &y'''(1)=(\cos 1\sin 1+\cosh 1\sinh 1)/(\cosh 2+\cos 2). \end{aligned} \quad (33)$$

The exact solution for the above problem is given by y(x) =0.25[1-2(sin1 sinh1 sinx sinhx + cos1 cosh1 cosx coshx)/(cosh2+cos2)]. The proposed method is tested on this problem where the domain [-1,1] is divided into 10 equal subintervals. Numerical results for this problem are shown in Table 4. The maximum absolute error obtained by the proposed method is 1.966953×10^{-6}.

Table 4. Numerical results for example 4

x	Exact Solution	Absolute error by the proposed method
-0.8	3.976926E-02	2.235174E-07
-0.6	7.498498E-02	2.533197E-07
-0.4	1.023106E-01	2.756715E-07
-0.2	1.195382E-01	8.344650E-07
0.0	1.254157E-01	1.966953E-06
0.2	1.195382E-01	1.668930E-06
0.4	1.023106E-01	1.944602E-06
0.6	7.498498E-02	1.773238E-06
0.8	3.976926E-02	1.031905E-06

Example 5.

Consider the nonlinear boundary value problem

$$y^{(8)}=e^{-x}y^{2}(x),\ 0<x<1 \tag{34}$$

subject to the boundary conditions

$$\begin{aligned} &y(0)=1, && y(1)=e,\\ &y'(0)=1, && y'(1)=e,\\ &y''(0)=1, && y''(1)=e,\\ &y'''(0)=1, && y'''(1)=e. \end{aligned} \tag{35}$$

The exact solution for the above problem is given by y(x)=exp(x). This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [23] as

$$y_{(n+1)}^{(8)}+[-2e^{-x}y_{(n)}]y_{(n+1)}=-e^{-x}y_{(n)}^{2},\ \text{for } n=0, 1, 2... \tag{36}$$

subject to the boundary conditions

$$\begin{aligned} &y_{(n+1)}(0)=1, && y_{(n+1)}(1) = e,\\ &y_{(n+1)}'(0)=1, && y_{(n+1)}'(1) = e,\\ &y_{(n+1)}''(0)=1, && y_{(n+1)}''(1) = e,\\ &y_{(n+1)}'''(0)=1, && y_{(n+1)}'''(1) = e. \end{aligned} \tag{37}$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (36). Numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is 5.340576×10^{-5}.

Table 5. Numerical results for example 5

x	Exact Solution	Absolute error by the proposed method
0.1	1.105171	2.264977E-06
0.2	1.221403	6.079674E-06
0.3	1.349859	3.480911E-05
0.4	1.491825	4.649162E-05
0.5	1.648721	3.528595E-05
0.6	1.822119	5.340576E-05
0.7	2.013753	1.978874E-05
0.8	2.225541	5.006790E-06
0.9	2.459603	3.123283E-05

Example 6.

Consider the nonlinear boundary value problem

$$y^{(8)}=7![e^{-8y}-2/(1+x)^8],\ 0<x<e^{1/2}-1 \tag{38}$$

subject to the boundary conditions

$$\begin{aligned} &y(0)=0, && y(e^{1/2}-1)=1/2,\\ &y'(0)=1, && y'(e^{1/2}-1)=1/[e^{1/2}],\\ &y''(0)=-1, && y''(e^{1/2}-1)=-1/e,\\ &y'''(0)=2, && y'''(e^{1/2}-1)=2/[e^{3/2}]. \end{aligned} \tag{39}$$

The exact solution for the above problem is given by y(x) = ln(1+x). This nonlinear boundary value problem is converted into a sequence of linear boundary value problems generated by quasilinearization technique [23] as

$$y_{(n+1)}^{(8)}+[8!e^{-8y(n)}]y_{(n+1)}=e^{-8y(n)}[8!y_{(n)}-7!]-\frac{(2)7!}{(1+x)^8},\quad \text{for } n=0, 1, 2... \tag{40}$$

subject to the boundary conditions

$$\begin{aligned} &y_{(n+1)}(0) = 0, && y_{(n+1)}(e^{1/2}-1) =1/2,\\ &y_{(n+1)}'(0) = 1, && y_{(n+1)}'(e^{1/2}-1) =1/[e^{1/2}],\\ &y_{(n+1)}''(0) = -1, && y_{(n+1)}''(e^{1/2}-1) = -1/e,\\ &y_{(n+1)}'''(0) = 2, && y_{(n+1)}'''(e^{1/2}-1) =2/[e^{3/2}]. \end{aligned} \tag{41}$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for y. The domain $[0, e^{1/2}-1]$ is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (40). Numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 2.184510×10^{-5}.

Table 6. Numerical results for example 6

x	Exact Solution	Absolute error by the proposed method
0.1	9.531018E-02	1.862645E-07
0.2	1.823216E-01	1.028180E-06
0.3	2.623643E-01	1.153350E-05
0.4	3.364722E-01	1.737475E-05
0.5	4.054651E-01	2.184510E-05
0.6	4.700036E-01	6.556511E-07

7. Conclusions

In collocation method, the number of basis functions in the approximation of the solution should match with the number of selected collocation points to get an accurate solution. In view of this, in this paper we have developed a collocation method with sextic B-splines as basis functions to solve eighth order boundary value problems. Here we have taken internal mesh points $x_1, x_2, \ldots, x_{n-2}$ as the selected collocation points. The sextic B-spline basis set, which has originally $n+6$ basis functions, has been redefined into a new set of basis functions which in number match with the number of selected collocation points. The proposed method is applied to solve several number of linear and non-linear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple and direct technique to solve a eighth order boundary value problem.

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