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Effect of Slip Parameter on the Flow of Viscous Fluid Past an Impervious Sphere

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Abstract: In this paper, the uniform flow of an incompressible, axi-symmetric, viscous fluid over a stationary impervious sphere with interface slip on its surface is considered. *Homotopy Analysis Method* (HAM) is used to solve the non linear momentum equations for stream function. To match with the uniform flow far away from the sphere, stream function is taken in terms of Gegenbauer polynomials. The solution obtained is found to be convergent and is seen to be in good agreement with the results available in literature. Drag acting on the sphere due to the flow and vorticity functions is found. For different values of the slip parameter, drag acting on the sphere is evaluated and the results are in good agreement with the available experimental data for the Reynolds numbers less than 50. Expansion of Gegenbauer polynomials and solution of the problem are obtained using MATHEMATICA.

Keywords: Slip condition; HAM; gegenbauer polynomials.

1. Introduction

In the classical problems of fluid flow past bodies, it is usual to apply no-slip condition on the surface of the bodies under considerations. But with the advent of miniature devices and experiments with rarified gases, it is observed that fluid slips on the surface of the bodies in the following situations.

i) When the fluid flow occurs in rarified gases at low density and low pressure. If Knudsen number, Kn (the ratio of mean free path to body size) is in the ranges from 0.01 to 130. (Knudsen [1], Millikan [2], Schaaf and Chambre [3], Sreekanth, [4], Hinds [5], Gad-el-Hak [6], and Abouzar Moshfegh [7]).
ii) When fluid flow occurs over the surfaces of a porous medium (Beavers and Joseph [8], Saffman [9]).
iii) In the latter stages of combustion, fuel droplets experience slip over the surface (Crowe [10]).
iv) When the size of the body under consideration is comparable with the mean free path of the fluid particles, (when Kn is less than 0.1) slip occurs. (Gabriel [11], Gravesen [12], Gad-el-Hak [13], Barber and Emerson [14], Luo and Pozrikidis [15]).
v) When water flows near a hydrophobic surface (a surface that repels water molecules), slip occurs. (Vinogradova [16], Tretheway and Meinhart [17], Neto Evans [18], Cottin-Bizonne [19]).

In these above situations, the Reynolds number for the flow ranges from small values 0.01 to

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moderately high values up to 120 (Niazmand and Anbarsooz [20]). The flow past bodies (many researchers considered the body as sphere) at moderate values of Reynolds numbers has been analyzed by singular perturbation methods or by numerical methods.

Stokes [21] was the first person to derive the expression for a drag acting on a sphere moving with uniform velocity. He neglected the nonlinear convective terms in the equations of motion. This assumption is known as “Stokesian” approximation. Whitehead [22] tried to improve the solution for velocity by using a straight forward perturbation scheme starting with stokes solution as the initial guess. He found that there exists no solution. The inability to extent stokes solution by the iterative scheme is known as white head's paradox. Oseen [23] showed that for Stokes solution, the ratio of inertial terms to viscous terms far from the sphere are of the order of local Reynolds numbers and hence Stokes solution cannot be taken as initial guess in White Head's iterative scheme. Oseen obtained the expression for drag in series of Reynolds number by linearizing the nonlinear convective terms with uniform flow. Later improvement in the solution was done by Goldstein [24], Shanks [25] and Kaplun and Lagersrom [26]. Proudman and Pearson [27] have established a rigorous mathematical method based on matched asymptotic expansion for singular perturbation solution. But still the solution thus obtained is valid only for small range of Reynolds numbers <5. Van Dyke [28], applied the series truncation method to get drag valid up to Reynolds numbers 10.

A detailed description of the methods for flow past a sphere and matching techniques can be found in the treatise ``Slow Viscous flow" of Langlois [29] and in the monograph by Michaelides [30]. Thus, the analytical solutions for drag are valid up to Reynolds numbers 10.

But the numerical methods developed by Dennis and Walker [31], yield solution which can match with the experimental data of Takaki [32] for Reynolds numbers up to 40. The Homotopy Analysis Method (HAM) developed by Liao [[33], [34]] gives drag for Reynolds number up to 40. All the above mentioned authors studied the flow past a sphere with no slip condition.

The slip condition was examined very long back by Navier [35] and Maxwell [36]. For creeping flow past a sphere, Basset [37] obtained the coefficient of drag C_d in terms of slip parameter s (Trostel number) as:

$$C_d = \frac{24}{Re}\left(\frac{s+2}{s+3}\right) \tag{1.1}$$

where $s = \dfrac{\beta a}{\mu}$ (defined as slip parameter or Trostal number). s is related to Tangential Momentum Accommodation Coefficient (TMAC) σ and Knudsen number Kn (Schaaf and Chambre [3], Trostel [38], Atefi [39]) by the formula

$$s = \frac{\sigma}{(2-\sigma)Kn} \tag{1.2}$$

where interfacial slip occurs, Knudsen number may vary from 0.1 to 10 for a wide range of Reynolds number 0.1<Re<100. For a small Reynolds number (Re<1), Keh and Shiau [40] obtained the solution by singular perturbation method and found the following result:

$$C_d = \frac{24}{Re}\alpha\left[1+\frac{3}{8}\alpha?Re+\frac{9}{40}\ \alpha)^2 Re^2 \log(Re)+O(Re)^2\right], \text{ where } \alpha = \left(\frac{s+2}{s+3}\right) \tag{1.3}$$

Michaelides and Feng [41] discussed the effect of slip on spherical viscous drop in an unsteady flow. Zhi-Gang Feng [42] obtained a correlation formula for drag valid for a wide range of Reynolds numbers. Feng et al [43] presented a comprehensive study on the drag with slip condition. He obtained a solution for stream function using singular perturbation approach by matching the inner and outer region solutions. This formula covers many special cases such as solid sphere with or without slip, inviscid bubbles and viscous droplets with or without slip. It is worth mentioning to refer the problems related to flow past a sphere with slip condition examined by Datta and Singhal [44], Michael Miksis and Stephen Davis [45], Datta and Deo [46].

In view of non availability of analytical solution for moderate Reynolds numbers, in the present paper, we attempt to obtain solution for uniform flow past a impervious sphere with slip condition at moderate Reynolds numbers using Homotopy Analysis Method (HAM).

2. Formulation of the problem

We consider a solid sphere of radius a held fixed in a steady uniform flow of a viscous fluid with free stream velocity U_0. A spherical coordinate system (r,θ,ϕ) with unit base vectors $(\overline{e_r},\overline{e_\theta},\overline{e_\phi})$ is taken with origin at the center of the sphere and Z-axis along the direction of uniform flow. The scale factors for the system are $h_1=1, h_2=R, h_3=R\sin\theta$. The flow is assumed to be laminar, in compressible and axi symmetric. Hence the flow is in the meridian plane of $\overline{e_r},\overline{e_\theta}$ and all physical quantities are independent of the toroidal coordinate ϕ.

The velocity vector is chosen in the form

$$\overline{Q}=U(R,\theta)\overline{e_r}+V(R,\theta)\overline{e_\theta} \tag{2.1}$$

In view of the incompressibility condition

$$div\overline{Q}=o \tag{2.2}$$

velocity can be written in terms of stream function ψ as

$$U(R,\theta)=\frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta};V(R,\theta)=\frac{-1}{R\sin\theta}\frac{\partial\Psi}{\partial R} \tag{2.3}$$

The equations of motion are given by

$$\rho\frac{d\overline{Q}}{dt}=\rho\Big(\frac{\partial\overline{Q}}{\partial t}+\overline{Q}\cdot\nabla_0\overline{Q}\Big)=-\nabla_0P+\mu\nabla_0^2\overline{Q} \tag{2.4}$$

where ∇_0 is dimensional gradient.

We introduce the following non dimensional scheme (with capitals for physical quantities and small letters for non-dimensional quantities):

$$R=ar,\overline{Q}=U_0\overline{q},U=U_0u,V=V_0v,\Psi=a^2U_0\psi,P=\rho U_0^2p,\nabla_0=\frac{1}{a}\nabla \tag{2.5}$$

and Reynolds number $Re=\dfrac{2a\rho U_0}{\mu}$.

Considering the steady flow, the equation (2.4) can be rewritten in the non dimensional form as:

$$\frac{Re}{2}\Big(\frac{1}{2}\nabla\overline{q}^{2}-\overline{q}\times(\nabla\times\overline{q})\Big)=-\nabla p-\nabla\times(\nabla\times\overline{q}) \tag{2.6}$$

Elimination pressure p from (2.6) we get

$$E^{4}\psi=\frac{Re}{2r^{2}}\Big[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}+\frac{2x}{1-x^{2}}\frac{\partial\psi}{\partial r}+\frac{2}{r}\frac{\partial\psi}{\partial x}\Big]E^{2}\psi \tag{2.7}$$

where $x=\cos\theta$ and the Stokes stream function operator E^2 is

$$E^{2}=\frac{\partial^{2}}{\partial r^{2}}+\frac{(1-x^{2})}{r^{2}}\frac{\partial^{2}}{\partial x^{2}}=\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r^{2}}\frac{\partial^{2}}{\partial\theta^{2}}-\frac{\cot\theta}{r^{2}}\frac{\partial}{\partial\theta}$$

The slip boundary condition assumes that the tangential velocity of the fluid relative to the solid at a point on a surface is proportional to the tangential stress acting at that point. The constant of proportionality between these two is termed as a coefficient of sliding friction. Thus the slip boundary condition as in Happel and Brenner [47] is given by

$$T_{r\theta}=\beta(q_{\theta}-V_{\theta})\ on\ r=a \tag{2.8a}$$

where q_θ is fluid velocity and V_θ is the velocity of the sphere along θ direction. β is coefficient of sliding friction (Here $V_\theta=0$ since the sphere is stationery).

In non dimensional form, this condition reduces to

$$\frac{\partial^{2}\psi}{\partial r^{2}}=(s+2)\frac{\partial\psi}{\partial r}\ on\ \ r=1\ \ with\ \ s=\frac{\beta a}{\mu} \tag{2.8b}$$

where s is non-dimensional slip parameter (It is also called as “Trostel number”). The case of $\beta\to\infty$ (i.e., $s\to\infty$) leads to the no-slip condition. For the uniform flow past a sphere, the conditions on flow variables in non dimensional form are

$$(i).\ u=0\ on\ r=1\ i.e.,\ \psi=0\ on\ r=1\ \ (impermeability\ condition) \tag{2.9a}$$

$$(ii).\ \ T_{r\theta}=sv\ \ on\ \ r=1\ \ i.e.,\ \ \frac{\partial^{2}\psi}{\partial r^{2}}=(s+2)\frac{\partial\psi}{\partial r}\ \ on\ \ r=1\ \ (slip\ \ Condition) \tag{2.9b}$$

$$(iii).\ lim\overline{q}=1\ i.e.,\ lim\,\psi=\frac{1}{2}r^{2}(1-x^{2})\ \ (uniform\ flow\ condition) \tag{2.9c}$$

The equation (2.7) can be put in the form

$$E^{2}\psi=-\Omega \tag{2.10}$$

and $$E^{2}\Omega=\frac{Re}{2r^{2}}\Big[\frac{\partial\psi}{\partial r}\frac{\partial\Omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial r}+\Big(\frac{2x}{1-x^{2}}\frac{\partial\psi}{\partial r}+\frac{2}{r}\frac{\partial\psi}{\partial x}\Big)\,\Omega\Big] \tag{2.11}$$

Using (2.9), the conditions on Ω can be taken in terms of ψ as:

$$\Omega=-\frac{\partial^{2}\psi}{\partial r^{2}}\ \ at\ \ r=1\ \ and\ \ \lim_{r\to\infty}\Omega=0. \tag{2.12}$$

Now the problem reduces to finding solution to (2.10) and (2.11) subject to the conditions (2.9) using Homotopy Analysis Method (HAM) which is being introduced in Section (3).

3. The basic idea of HAM

Consider a non linear differential equation

$N[f(x,t)] = 0$ (3.1)

where N is the differential operator(with linear and non linear terms), f(x, t)is the unknown function to be solved for spatial variable x and temporal variable t. The main aim of HAM is that instead of solving (3.1), a *Homotopy* equation with a homotopy parameter λ, can be constructed for a homotopy function F such that

$$(1-\lambda)\ L[F(x,t:\lambda)-f_0(x,t)] = \lambda\, H\, \hbar\, N[F(x,t;\lambda)],\quad \lambda\varepsilon[0,1] \tag{3.2}$$

The equation (3.2) is called zeroth order deformation equation of (3.1). The homotopy equation reduces to simple linear equation when λ=0 and yields the original differential equation (3.1) when λ=1. This can be accomplished by the following function F(x, t; λ), the mapping function or homotopy function. F(x, t; λ) smoothly changes from $f_0(x,t)$, an initial estimate of *f(x,t)* to final (or target) solution f(x, t). $\hbar$ is a non-zero auxiliary parameter(convergence control parameter). H(x, t) is a non-zero auxiliary real function. L is a linear operator. N[f] is the given differential equation.

The mapping function F is to satisfy the properties that

$(i).\ F(x,t;0) = f_0(x,t)$

$(ii).\ F(x,t;1) = f(x,t)$ *and*

(iii). $N[F(x,t;1)] = 0$ i.e., $N[f(x,t)] = 0$.

As the embedding *homotopy* parameter λ varies from 0 to 1, F(x, t; λ) maps continuously from the initial estimate of $f_0(x,t)$ to the final exact solution f(x, t). By Maclaurin's Theorem, F(x, t; λ) can be expanded with respect to the embedding (*homotopy*) parameter λ as

$$F(x,t;\lambda) = f_0(x,t) + \sum_{m=1}^{\infty} f_m(x,t)\lambda^m \tag{3.3}$$

$$\text{where}\quad f_m(x,t) = \frac{1}{m!}\frac{\partial^m F(x,t;\lambda)}{\partial \lambda^m}\Big|_{\lambda=0} \tag{3.4}$$

Differentiating the zeroth-order deformation equation (3.1) m-times with respect to λ at λ=0 and the dividing it by m!, we get the following m^{th} -order deformation equation

$$L[f_m(x,t) - \chi_m f_{m-1}(x,t)] = \hbar\ H\ R_m(x,t) \tag{3.5}$$

$$\text{where}\quad \chi_m = \begin{cases} 0 & if\ \ m = 1 \\ 1 & if\ \ m > 1 \end{cases}$$

$$R_m(x,t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[F(x,t;\lambda)]}{\partial \lambda^{m-1}} \Big|_{\lambda=0} \tag{3.6}$$

If the series converges, we have the exact solution at λ=1

$$F(x,t,1) = f(x,t) = f_0(x,t) + \sum_{m=1}^{\infty} f_m(x,t) \tag{3.7}$$

In this method, Linear operator L, initial approximation $f_0(x,t)$ and auxiliary real function H(x, t) can be chosen conveniently. The initial guess f_0 and the Linear Operator L can be chosen (Liao [48], Liao [49]) according to the rule of solution expression and rule of solution existence. The auxiliary function H can be chosen by the rule of coefficient of ergodicity. The convergence control parameter h can be chosen within an interval in which h-curves are parallel to h-axis (F(x, t, h) at a fixed large value of m at fixed values of $x = x_0, t = t_0$ gives a polynomial in h, this is called h-curve). In general, a physical quantity (drag, volumetric flow rate, skin friction etc.,) is evaluated and the value of h will be decided.

The auxiliary (convergence control) parameter $\hbar$ provides us a convenient way to control and adjust the rate and region of the convergence. More information about properties of h-curves of HAM, a study on the convergence of HAM, the essence of homotopy analysis method can be found in Abbasbandy et. al., [50], Zaid odibat [51], Cheng-Shi Liu [52] respectively. Cheng-Shi Liu [53] considered HAM as a generalized Taylor series.

4. Solution of the problem

The homotopy functions ψ_h and Ω_h for the stream function ψ, Swirl Ω are expanded in power series for λ as:

$$\Omega_h(r,x,\lambda) = \Omega_0(r,x) + \Omega_1(r,x)\lambda + \Omega_2(r,x)\lambda^2 + \cdots. \tag{4.1}$$

$$\Omega_h(r,x,\lambda) = \Omega_0(r,x) + \Omega_1(r,x)\lambda + \Omega_2(r,x)\lambda^2 + \cdots. \tag{4.2}$$

It is assumed that the series (4.1) and (4.2) converge to stream function ψ and swirl Ω respectively at the *homotopy* parameter λ=1.

$$\psi_h(r,x,1) = \psi = \sum_{i=0}^{\infty} \psi_i \tag{4.3a}$$

$$\Omega_h(r,x,1) = \Omega = \sum_{i=0}^{\infty} \Omega_i \tag{4.3b}$$

The *homotopy* equations for equations (2.10) and (2.11) are constructed such that when λ=0 initial solution $\Omega_h = \Omega_0$ is obtained and when λ=1, the exact solution $\Omega_h = \Omega$ is obtained.

$$E^2\psi_h = -\Omega_h \tag{4.4}$$

with

$$(1-\lambda)E^2(\Omega_h - \Omega_0) = \lambda h H\Big(E^2\Omega_h - \frac{Re}{2r^2}\Big[\frac{\partial \psi_h}{\partial r}\frac{\partial \Omega_h}{\partial x} - \frac{\partial \psi_h}{\partial x}\frac{\partial \Omega_h}{\partial r} + \frac{2x\Omega_h}{1-x^2}\frac{\partial \psi_h}{\partial r} + \frac{2\Omega_h}{r}\frac{\partial \psi_h}{\partial x}\Big]\Big) \tag{4.5}$$

These equations (4.4) and (4.5) are called the *zeroth order deformation*. Substituting (4.1) and (4.2) in (4.4) and (4.5) and collecting the coefficient λ^m, the m^{th} order deformation equations are obtained as follows:

$$E^2\Omega_1 = hH\left(E^2\Omega_0 - \frac{Re}{2r^2}\left[\frac{\partial\psi_0}{\partial r}\frac{\partial\Omega_0}{\partial x} - \frac{\partial\psi_0}{\partial x}\frac{\partial\Omega_0}{\partial r} + \frac{2x\Omega_0}{1-x^2}\frac{\partial\psi_0}{\partial r} + \frac{2\Omega_0}{r}\frac{\partial\psi_0}{\partial x}\right]\right) \quad for\ m=1 \tag{4.6}$$

$$E^2\Omega_m = E^2\Omega_n + hH\left(E^2\Omega_n - \frac{Re}{2r^2}\left[\frac{\partial\psi_n}{\partial r}*\frac{\partial\Omega_n}{\partial x} - \frac{\partial\psi_n}{\partial x}*\frac{\partial\Omega_n}{\partial r} + \frac{2x\Omega_n}{1-x^2}*\frac{\partial\psi_n}{\partial r} + \frac{2\Omega_n}{r}*\frac{\partial\psi_n}{\partial x}\right]\right) \quad for\ m\geq 2 \tag{4.7}$$

and $E^2\psi_m + \Omega_m = 0 \quad for\ all \quad m \geq 0$ (4.8)

where $n = m-1 \quad and \quad \psi_n * \Omega_n = convolution \quad sum = \sum_{k=0}^{n}\psi_k\Omega_{n-k}$.

We choose the initial guess functions ψ_0 and Ω_0 in such a way that ψ_0, Ω_0 satisfies the boundary conditions(2.9) and

$E^4\psi_0 = 0$ and $E^2\psi_0 = \Omega_0$

Hence we have

$$\psi_0 = \frac{1}{2(s+2)}\left[2(s+2)r^2 - (3s+4)r + \frac{s}{r}\right] G_2(x) = f_{02}(r)\ G_2(x) \tag{4.9}$$

$$\Omega_0 = -\left[\frac{3s+4}{s+2}\right]\frac{1}{r}\ G_2(x) = g_{02}(r)\ G_2(x) \tag{4.10}$$

where $G_2(x) = \frac{1}{2}(1-x^2)$ =Gegenbauer polynomials of order 2. Following Happel and Brenner [47], the solutions for ψ_m and Ω_m of (4.6)-(4.8) are assumed in the following form, containing Gegenbauer polynomials as base functions.

$$\psi_m(r,x) = \sum_{l=2}^{m+2} f_{m,l}(r)G_l(x) \tag{4.11a}$$

$$\Omega_m(r,x) = \sum_{l=2}^{m+2} g_{m,l}(r)G_l(x) \tag{4.11b}$$

To match the uniform flow far from the sphere the factor $(1-x^2)$ is to be retained. Hence summation starts from $l=2$, since all $G_l(x), l\geq 2$ polynomials of order l contain a factor $(1-x^2)$. The form of the functions in (4.11(a), 4.11(b)) yields the following simplifications:

$$E^2\psi_m = \sum_{l=2}^{m+2} D_l^2 f_{m,l}(r)G_l(x) \tag{4.12}$$

$$E^2\Omega_m = \sum_{l=2}^{m+2} D_l^2 g_{m,l}(r)G_l(x) \tag{4.13}$$

where $D_l^2 = \frac{d^2}{dr^2} - \frac{l(l-1)}{r^2}$

The products of Gegenbauer polynomials in the above equations (4.6), (4.7) can be expressed in terms of $G_l(x)$ as follows

$$G_i(x)G_j^{'}(x) = \sum_{l=2}^{i+j-1} a_{ijl} G_l(x) \tag{4.14}$$

$$\frac{x}{1-x^2} G_i(x)G_j(x) = \sum_{l=2}^{i+j-1} b_{ijl} G_l(x) \tag{4.15}$$

Now substituting the expressions (4.12)-(4.13) in the m^{th} order deformation equation (4.6) and collecting coefficient of $G_l(x)$ on both sides, we get:

$$D_l^2 g_{m,l}(r) = (\chi_m + hH) D_l^2 g_{m-1,l}(r) - \frac{hReH}{2r^2} J_{ml}(r) \tag{4.16a}$$

$$D_l^2 f_{n,l}(r) = -g_{n,l} l = 2,3,........m+2 \text{ , n=m-1} \tag{4.16b}$$

$$J_{m,l} = \sum_{i=2}^{k+2} \sum_{j=2}^{m-k+1} \left(a_{ijl} f'_{n,i} * g_{n,j} - a_{jil} f_{n,i} * g'_{n,j} + 2b_{ijl} f'_{n,i} * g_{n,j} + \frac{2}{r} a_{jil} f_{n,i} * g_{n,j} \right) \tag{4.16c}$$

where n=m-1 and $\chi_m = \begin{cases} 0, & if\ m = 1 \\ 1, & if\ m > 1 \end{cases}$

$f_{ni} * g_{nj} = convolution\ sum = \sum_{k=0}^{n} f_{ki} g_{kj},\ n = m-1$.

In the above equations (4.16), $g_{m,l}(r) = 0\ \ if\ \ l \geq m+3$.

The equations (4.16) are to be solved under the boundary conditions:

$$f_{m,l}(1) = f_{m,l}^{"}(1) - (s+2) f_{m,l}^{"}(1) = 0 \quad and \quad \lim_{r\to\infty} f_{m,l} = 0,$$

$$g_{m,l}(1) = -f_{m,l}^{"}(1) \ \ and \ \ \lim_{r\to\infty} g_{m,l} = 0 \quad m \geq 1,\ \ l \geq 2 \tag{4.17}$$

Now we have to choose the auxiliary function H in order to get the convergent solution. By uniform flow condition, as $r \to \infty$ we should have $f_{m,l} \to r^a$, $a < 2$, $g_{m,l} \to constant$ so that velocity tends to zero at a distance far away from the sphere.

This condition is satisfied when the auxiliary function $H = \frac{1}{r^\sigma}\ if\ \ \sigma > 0$. Starting from initial guess as in (4.9) and (4.10), the first order deformation equations are obtained from (4.16) for m=1, l=2:

$$D_2^2 g_{1,2}(r) = \frac{h}{r^\sigma} D_2^2 g_{0,2}(r) - \frac{hRe}{2r^{2+\sigma}} J_{1,2} = 0 \tag{4.18a}$$

and $D_2^2 f_{1,2}(r) = -g_{1,2}(r)$ (4.18b)

Since $f_{1,2}(r)$ satisfies homogeneous boundary conditions, we have

$$f_{1,2}(r) = g_{1,2}(r) = 0$$

For m=1, l =3, the equation (4.16) gives

$$D_3^2 g_{1,3} = -\frac{hRe}{2r^{2+\sigma}} J_{1,3} = \frac{9hRe}{(3+s)(6+2s)r^{\sigma}} \left(\frac{(6+s^2+5s)}{r^2} - \frac{6(1+s+\frac{s^2}{4})}{r^3} + \frac{s(1+\frac{s}{2})}{r^5} \right)$$

and $D_3^2 f_{1,3}(r) = -g_{1,3}$

If we take $\sigma = 0$, we get

$$f_{1,3}(r) = \frac{A}{r^2} + \frac{9hRe}{(3+s)^2}(r^2(\frac{1}{8}+\frac{5s}{48}+\frac{s^2}{48}) + r(-\frac{1}{8}-\frac{s}{8}) - \frac{1}{r}(-\frac{s}{48}-\frac{s^2}{96}) + \frac{1}{r^2}(\frac{8s+6s^2+s^3}{96(5+s)}) + (-\frac{s^2}{32}+\frac{12s+8s^2+s^3 96(5+s)}{96(5+s)})) \tag{4.19}$$

This solution does not give a vanishing velocity as $r \to \infty$. Hence this solution is not feasible. This non existence of solution is known as Whitehead's paradox [Langlois [29]]. Hence in $H = \frac{1}{r^{\sigma}}$ we have to choose $\sigma > 0$.

Here we have obtained solutions for the equation (4.16) by taking $H = \frac{1}{r^{\sigma}}$ for the cases $\sigma = 1$, $\sigma = 3/2$, $\sigma = 2$. We observed that the solution converges more rapidly as σ values are increasing. The equation (4.16) can be taken as

$$D_l^2 g_{ml} = RHS(r) = (\chi_m + \frac{h}{r^{\sigma}}) D_l^2 g_{m-1,l} - \frac{h\ Re}{2\ r^{2+\sigma}} J_{m,l} \tag{4.20a}$$

The solution of this equation is obtained by the method of variation of parameters as below:

$$g_{nl} = a_1 r^{1-l} + r^l \int \frac{RHS(r) r^{1-l}}{2l-1} dr - r^{1-l} \int \frac{RHS(r) r^l}{2l-1} dr \tag{4.20b}$$

then similarly solution for $f_{n,l}$ is found for the equation,

$$D_l^2 f_{nl} = -g_{nl} \tag{4.20c}$$

as follows $f_{nl} = a_2 r^{1-l} - r^l \int \frac{(g_{nl}) r^{1-l}}{2l-1} dr + r^{1-l} \int \frac{(g_{n-l}) r^l}{2l-1} dr.$ (4.20d)

The constants a_1 and a_2 are found from the conditions (4.17).

After getting f_{nl}, the stream function ψ_n at the n^{th} iteration is obtained from (4.11) and as n (number of iterations) increases the exact solution for stream function ψ is obtained at λ=1 from (4.3). Using Mathematica 7, the solution is obtained up to 11th order of approximation. After 11th order of approximation, it takes very long computational time nearly 6 hours for each order of iteration. For σ=1 our solution exactly matches with that of Liao [33], Liao [34] for $s \to \infty$ (which gives the solution with no-slip condition). We have plotted the stream function and vorticity function for different values of slip parameter s at different values of Reynolds number Re.

5. Drag on the sphere

The drag on a body submerged in a stream arises from two sources: the shear stress and the pressure distribution at the surface of the body. The drag due to the shear stress is called the skin friction and the part due to the pressure distribution is called the form drag. For a flat body the drag is fully due to shear stress, for a blunt body most of the drag is due to the pressure distribution.

Force on area element ds in the direction of flow= drag on the area element ds of the sphere is $=(T_{11}\cos\theta - T_{21}\sin\theta)ds$

The Drag due to the flow on the entire sphere is given by

$$Drag = D = 2\pi a^2 \int_0^{\pi} (T_{11}\cos\theta - T_{21}\sin\theta)\big|_{r=a} \sin\theta \quad d\theta \tag{5.1}$$

where $T_{11} = -P + 2\mu \dfrac{\partial U}{\partial R}$ *and* $T_{21} = \mu R \dfrac{\partial}{\partial R}(\dfrac{V}{R}) + \dfrac{\mu}{R}\dfrac{\partial U}{\partial \theta}$ (5.2)

Substituting (5.2) and (2.8) in (5.1) and expressing it in non dimensional form we have

$$D = 2\pi\mu a\, U_0 \int_{-1}^{1} \left[(-p + 2\frac{\partial u}{\partial r})x - s.v\sqrt{1-x^2} \right] \Big| r = a \ dx \ with \ x = \cos\theta \tag{5.3}$$

From (2.6), equation for pressure in the direction of $\overline{e_\theta}$ is given by

$$\frac{\partial p}{\partial \theta} = -\frac{Re}{4}\frac{\partial q^2}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial E^2(\psi)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\frac{E^2\psi}{h_3} \tag{5.4}$$

Following Chester and Breach [54], integrating (5.4) w.r.t θ at r=1 we get

$$p - p_0 = -\frac{Re}{4}(q^2(\theta) - q^2(0)) - \int_0^{\theta} \frac{\partial}{\partial r}(\frac{E^2\psi}{\sin\theta})d\theta \tag{5.5}$$

We observe that velocity q=0 on (the line of axi-symmetry) θ=0 and $q^2 = v^2$ on r=1.

Substituting (4.11) in this equation (5.5) and after simplification we get

$$p - p_0 = \frac{Re}{4}v^2 + \sum_{k=0}^{m}\sum_{l=2}^{m+2}\left[f'_{k,l}(1) - \frac{f'''_{k,l}(1)}{l(l-1)} \right](G'_l(x) + 1) \tag{5.6}$$

Substituting (5.6) in (5.3) we get

$$Drag = 2\pi\mu a\, U_0 \int_{-1}^{1} [x\frac{Re}{4}v^2 - \sum_{k=0}^{m}\sum_{l=2}^{m+2}\left[3f'_{k,l}(1) - \frac{f'''_{k,l}(1)}{l(l-1)} \right](xG'_l(x)) + \sum_{k=0}^{m}\sum_{l=2}^{m+2} sf'_{k,l}(1)\, G_l(x)]dx \tag{5.7}$$

The first term in (5.7), can be simplified by using (2.3) in non dimensional form, as

$$\int_{-1}^{1} (xv^2)\big|_{r=1} dx = \int_{-1}^{1} \frac{x}{1-x^2}(\sum_{k=2}^{m+1} F_k(1)G_k(x))^2 dx \tag{5.8}$$

where $F_k(1) = \sum_{l=k}^{m+2} f'_{l-2,k}(1),\ 2 \le k \le m+2$ (5.9)

Using the following properties (5.7), can be simplified to get drag.

$$\int_{-1}^{1}\frac{xG_mG_n}{1-x^2}dx=\begin{cases}\dfrac{2}{m(4m^2-1)} & if\ \ m+1=n\\ \dfrac{2}{(m-1)(4(m-1)^2-1)} & if\ \ m-1=n\end{cases}$$

$$\int_{-1}^{1}xG_l'(x)dx=\begin{cases}\dfrac{-2}{3} & if\ l=2\\ 0 & if\ l>2\end{cases}\quad \text{and}\quad \int_{-1}^{1}G_l(x)dx=\begin{cases}\dfrac{2}{3} & if\ l=2\\ 0 & if\ l>2\end{cases}$$

$$Drag=2\pi\mu a\,U_0\left[Re\sum_{i=1}^{m}\frac{F_{i+1}(1)F_{i+2}(1)}{(i+1)(2i+1)(2i+3)}+\frac{1}{3}\sum_{k=0}^{m}\left[\frac{(2s+6)}{(s+2)}f''_{k,2}(1)-f'''_{k,2}(1)\right]\right] \tag{5.10}$$

where $F_k(1)$ is defined in (5.9).

Non-dimensional drag Coefficient C_d is defined as

$$C_d=\frac{Drag}{\frac{1}{2}\rho U_0^2\pi a^2}=\frac{8}{Re}\sum_{m=0}^{13}\left[Re\sum_{i=1}^{m}\frac{F_{i+1}(1)F_{i+2}(1)}{(i+1)(2i+1)(2i+3)}+\frac{1}{3}\sum_{k=0}^{m}\left[\frac{(2s+6)}{(s+2)}f''_{k,2}(1)\text{-}f'''_{k,2}(1)\right]\right]$$

$$C_d=\frac{Drag}{\frac{1}{2}\rho U_0^2\pi a^2}=\frac{8}{Re}\sum_{m=0}^{13}\left[Re\sum_{i=1}^{m}\frac{F_{i+1}(1)F_{i+2}(1)}{(i+1)(2i+1)(2i+3)}+\frac{1}{3}\left[\frac{(2s+6)}{(s+2)}F_2'(1)\text{-}F_2''(1)\right]\right] \tag{5.11}$$

In the above equation (5.11) the upper limit is taken as 13, since we have taken up to m=13th order deformation

For m=0, $C_d=\frac{8}{Re}\left[\frac{2s+6}{s+2}f''_{02}(1)-f'''_{02}(1)\right]$

This gives the formula for Stokes flow with slip condition. Now as $s\to\infty$, we get $drag=\frac{24}{Re}$ which matches with the drag with no-slip condition for the Stokes flow.

6. Results and discussions

In the classical problem of flow past a sphere, it is a challenge to obtain an analytical solution valid for moderate Reynolds numbers. Recently, Liao [33] obtained an analytical solution by applying Homotopy Analysis Method. He observed that his method gives coefficient of drag (C_d) for Re≤30 by computing up to 10 iterations. He did not discuss the flow pattern. In our present paper, we obtain solution for flow past a sphere, under tangential slip condition, valid up to Re≤50 and up to 13 iterations. As far as the authors know, there was no contribution to analytical solution of the problem presented here for Re≤50 and here we present such a case. Our results include the previous studies.

The stream line pattern for the uniform flow past a sphere is obtained by solving equations (2.10) and (2.11) under the slip and uniform flow at infinity conditions. The solution of these non-linear equations can be obtained by forming homotopy equations or zeroth order deformation equations (4.4) and (4.5) which reduce to (2.10) and (2.11), when the homotopy parameter $\lambda \to 1$ these equations give the exact solution. When the parameter $\lambda \to 0$, the homotopy equations (4.4) and (4.5) reduce to simple linear equations and give the initial guess for the solution. The solution of the zeroth order deformation equations are made fast converging by choosing auxiliary function H appropriately. The homotopy equations (4.4) and (4.5) are expanded in base functions of Gegenbaur polynomials to get mth order deformation equation (4.16) with boundary conditions as mentioned in (4.17).

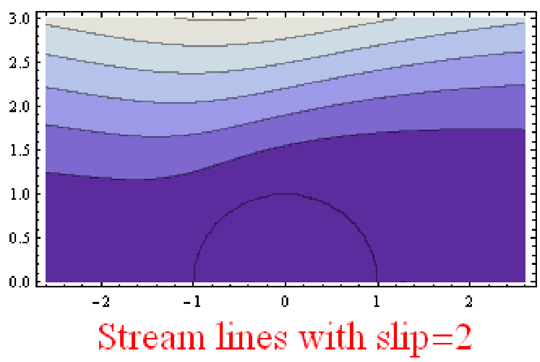
The equations (4.16) or (4.20) are solved, by taking $H = \frac{1}{r^{\sigma}}$ for the cases σ=1, 3/2 and 2. These different cases for σ are considered to check the convergence of the method for lower number of approximations.

Case σ=0: The solution for n=1 is presented in (4.19). It can be noted that the solution will not satisfy the uniform stream condition. Hence *σ must be positive to get the solution which satisfies the regularity condition.*

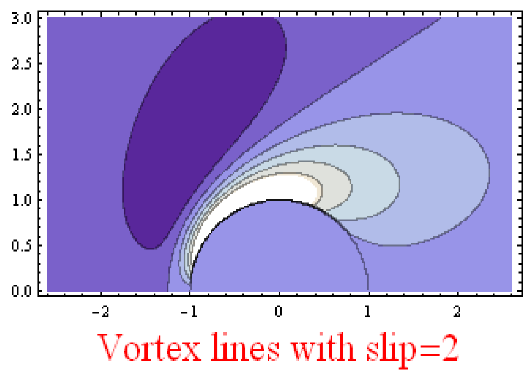
Case σ=1: The solution in this case is exactly matching with that of the solution obtained by Liao [34] for large values of slip parameter s i.e., for the case of no-slip boundary condition.

Case σ=2: The stream lines and vorticity lines are drawn for this case and values of drag are computed up to 13 iterations. The values of coefficient of drag (C_d) are matching for Re≤50 to the results of Takaki [32] for the case $s \to \infty$ and to the numerical results of Atefi [39] and Zhi-Gang Feng [42] for general slip parameter. The values of C_d are within 10% error when σ=3/2 and within 8% error for σ=2.

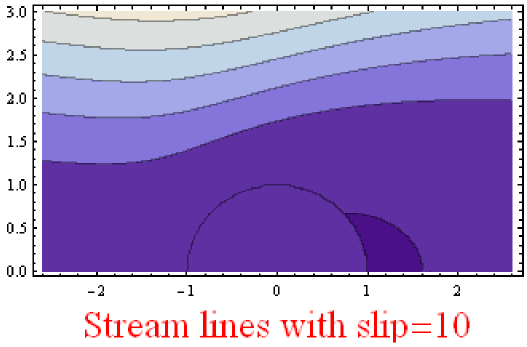
In Figure 1, the stream line pattern and vorticity lines are shown at different values of slip parameter at Re=250. The bright region shows high positive values and dark region shows low and negative values of stream lines/vorticity lines. As s increases (when no slip condition is taken), we observe that the region behind the sphere becomes more and more dark showing wakes and low values of stream lines(/vorticity lines). It is observed that as slip parameter $s \to 0$, the size of wake formation reduces and finally disappears. This is the condition for perfect slip. In the figures for vorticity, the brighter region near the sphere increases as slip values s are decreasing i.e., values of the vorticity lines near the sphere are having higher positive values and hence indicate strong vorticity zones. It is observed that as slip increases we find the formation of wakes behind the sphere showing the case of no-slip condition. At low values of slip <10, for the stream line pattern, the wakes behind the sphere are not observed (i.e., for perfect slip, no wakes are formed). From these figures of (1) (as $s \to \infty$ as we near to no slip condition), wakes are observed behind the sphere at Re =250.



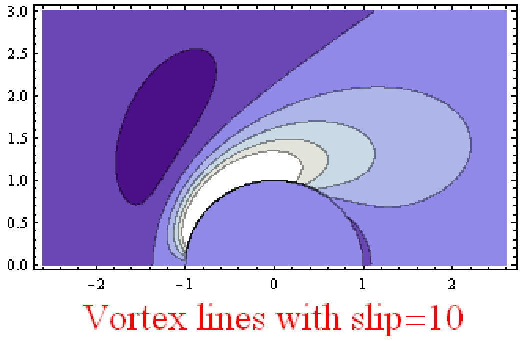
(a) Slip=2



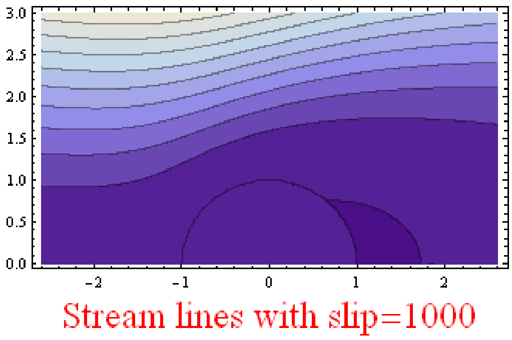
(b) Slip=2



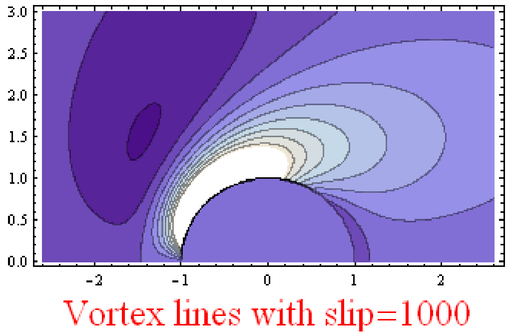
(c) Slip= 10



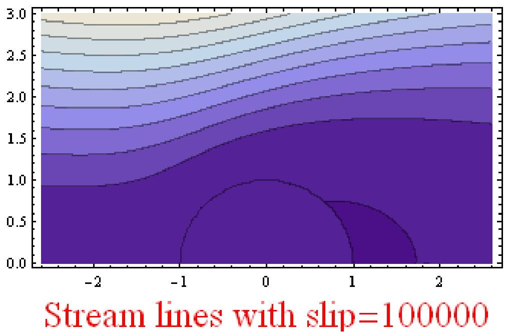
(d) Slip= 10



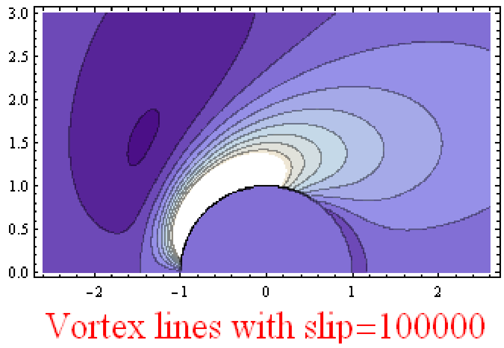
(e) Slip= 1000



(f) Slip= 1000



(g) Slip= 100000



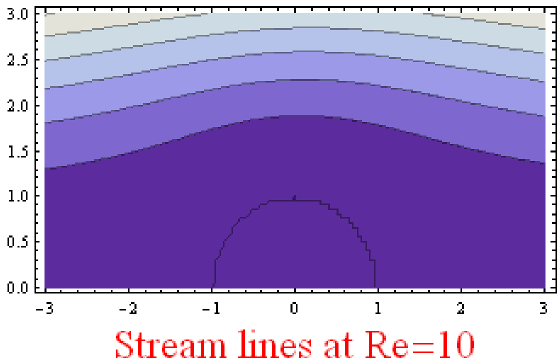
(h) Slip= 100000

Figure 1. (a), (c), (e), (g) Are stream lines and (b), (d), (f), (h) are vorticity lines at different slip parameter values for σ=2, Re=250(fixed) at 9th iteration

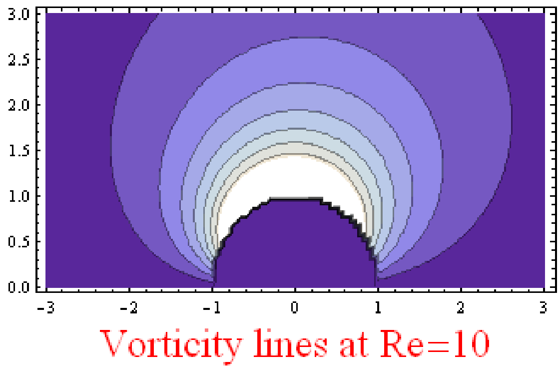
In Figure 2, the stream line pattern and vorticity line pattern are shown for different values of Reynolds number (Re) at 9^{th} iteration. As Re increases, occurrence of low values of vorticity in the direction of flow near the sphere and formation of wakes behind the sphere are observed. This observation is not clear up to 9^{th} iteration (these figures are not shown). As the iterations are increased, the vorticity diffusion at large Reynolds number is clear.

In order to get the values of C_d and stream lines pattern, the choice of h plays a very important role. To find the range of h values for convergence of the solution, first we have drawn curves for $\frac{\partial^2\psi}{\partial r^2}$ values verses h at r=1, $x=\frac{1}{2}$ (x is fixed randomly). The range of h values for which $\frac{\partial^2\psi}{\partial r^2}$ is constant is found (i.e., where the curve is parallel to h axis). A proper h value is chosen from this range based on higher rate of convergence. For that value of h graphs for C_d and stream lines are drawn.

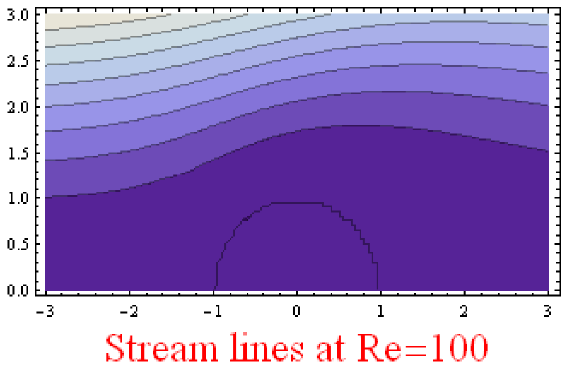
In the expression for drag, in (Equation (5.11) as $s\to\infty$, we notice that $F_k(1)=0$ and the formula for coefficient of drag at no slip condition agrees with the formula given by Liao[34]. In Figure 3, drag coefficient (C_d) versus Reynolds number (Re) is shown with high slip parameter s = 1000000. When $\sigma\geq 1$ and as the iterations are increasing, the results are nearer to the experimental values for Re≈50. These values are matching with the results of Atefi [39].



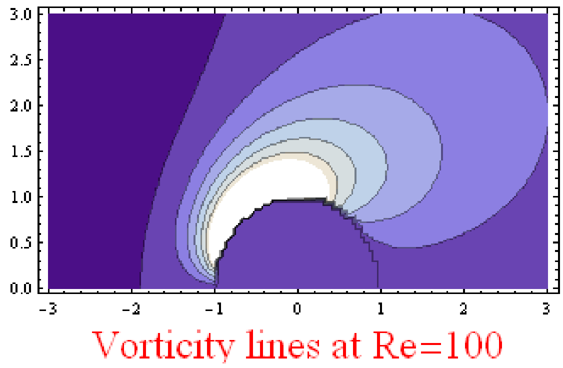
(a) Re= 10



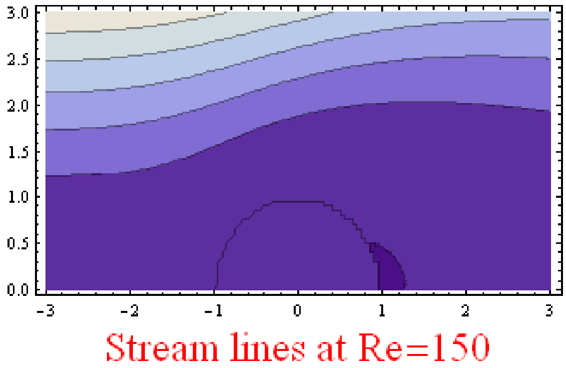
(b) Re= 10



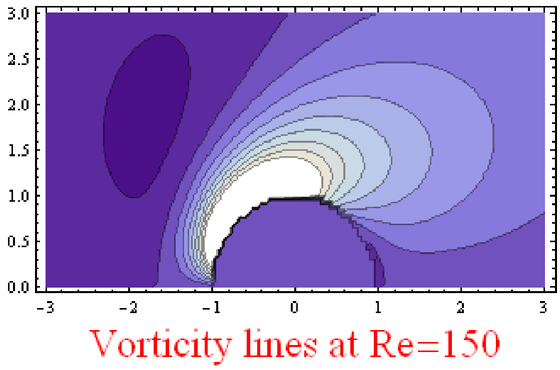
(c) Re=100



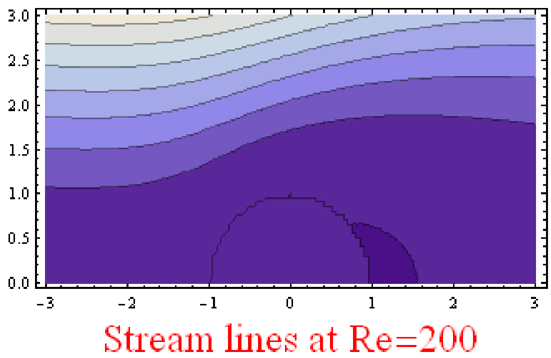
(d) Re=100



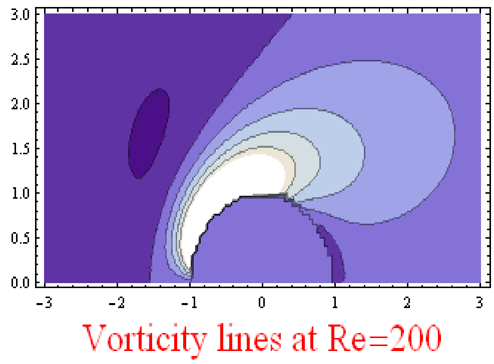
(e) Re=150



(f) Re=150



(g) Re=200



(h) Re=200

Figure 2. (a), (c), (e), (g) Are stream lines and (b), (d), (f), (h) are vorticity lines for different Re values with slip=100000 σ=2, at 9th iteration

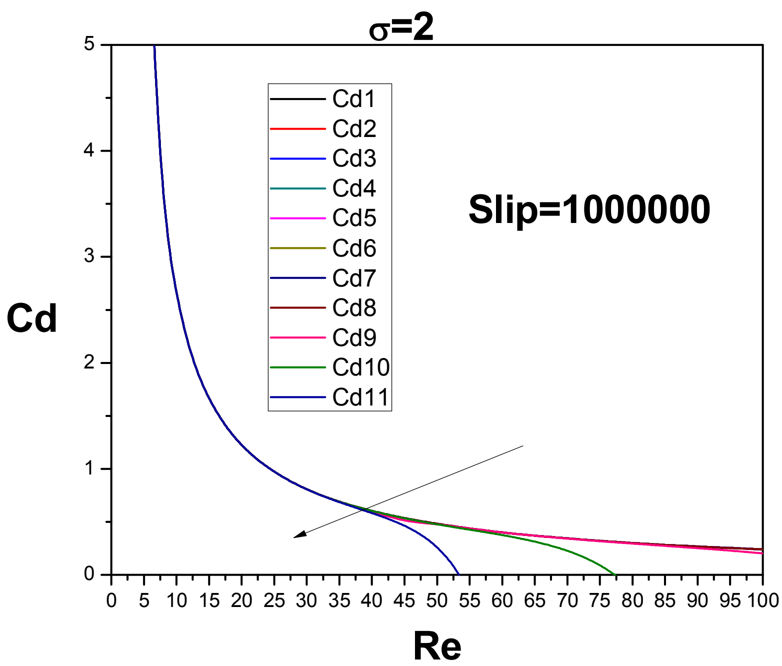


Figure 3. Cd graph with σ=2 at slip parameter value S=1000000

When slip parameter $s \le 20$ and Reynolds number $\mathrm{Re} \le 1$ our results are fairly in good agreement with the results reported by Zhi-Gang Feng [42](In Feng $Re = \dfrac{aU\rho}{\mu}$ and we took $Re = \dfrac{2aU\rho}{\mu}$). These results are shown in the Table 1 at 11^{th} iteration for σ=1, the error with the values of Feng[42] is less than 14.5% and in Table 2 for σ the error is le than 14%. From these tables, we observe that as σ increases, the error between our results and of Feng decreases. But we have not gone beyond σ>2. The same values are shown in graphical form in Figure 4 (a) and 4 (b). The values of C_d for slip parameter $s \le 20$ and $Re \le 1$ and for slip ≥ 1000 and $\mathrm{Re} \le 50$, our results are in good agreement with the available results of Feng [42] and Atefi [39], respectively.

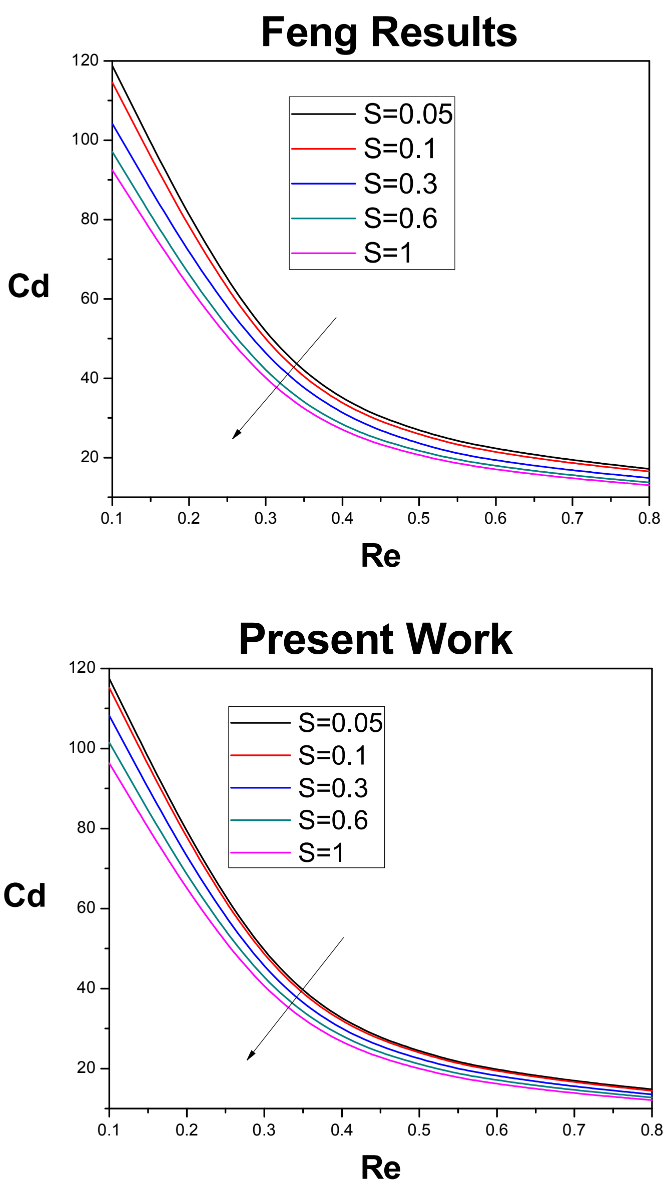


Figure 4. Cd graphs at 11th iteration for different SLIP values. here in the graphs Si indicates coefficient of drag in ith slip value

Table 1. Comparison of drag coefficients for different reynolds number values Vs different slip parameter values at the 11th iteration with σ=1. The values of slip parameter s in our study are equal to reciprocal values of slip parameter of Zhi-Gang Feng [42]

Re	Slip= 20			Slip=10			Slip=3.33		
	Present	Feng	Error%	Present	Feng	Error%	Present	Feng	Error%
0.1	117.46	118.8	1.128	115.186	114.6	0.509	108.126	104.2	3.631
0.3	39.15	41.62	5.935	38.39	40.08	4.217	36.04	37.85	4.782
0.5	23.499	26.04	9.758	23.08	25.05	7.86	21.63	22.65	4.23
0.6	19.58	22.12	11.48	19.20	21.26	9.69	18.02	19.2	6.146
0.7	16.78	19.3	13.05	16.46	18.54	11.21	15.45	16.73	7.651
0.8	14.692	17.17	14.43	14.407	16.5	12.7	13.52	14.87	9.07

Re	Slip=1.66			Slip=1		
	Present	Feng	Error%	Present	Feng	Error%
0.1	101.48	97.1	4.316	96.29	92.5	3.936
0.3	33.83	33.75	0.236	32.10	32.12	0.062
0.5	20.30	21	3.333	19.26	19.96	3.507
0.6	16.92	17.79	4.89	16.05	16.9	5.03
0.7	14.5	15.48	6.331	13.76	14.7	6.395
0.8	12.69	13.75	7.709	12.04	13.05	7.739

Table 2. Comparison of drag coefficients for different reynolds number values Vs different slip parameter values at the 11th iteration with σ=2. The values of slip parameter s in our study are equal to reciprocal values of slip parameter of Zhi-Gang Feng [42]

Re	Slip= 20			Slip= 10			Slip=3.33		
	Present	Feng	Error%	Present	Feng	Error%	Present	Feng	Error%
0.1	117.47	118.8	1.12	115.19	114.6	0.512	108.23	104.2	3.72
0.3	39.18	41.62	5.863	38.42	40.08	4.142	36.06	37.85	4.72
0.5	23.53	26.04	9.639	23.08	25.05	7.864	21.66	22.65	4.37
0.6	19.62	22.12	11.302	19.25	21.26	9.45	18.06	19.2	5.93
0.7	16.83	19.3	12.79	16.52	18.54	10.89	15.49	16.73	7.41
0.8	14.78	17.17	13.92	14.47	16.5	12.3	13.57	14.87	8.74

Re	Slip=1.66			Slip= 1		
	Present	Feng	Error%	Present	Feng	Error%
0.1	101.49	97.1	4.32	96.30	92.5	3.94
0.3	33.85	33.75	0.29	32.11	32.12	0.03
0.5	20.33	21	3.19	19.28	19.96	3.4
0.6	16.95	17.79	4.72	16.08	16.9	4.85
0.7	14.54	15.48	6.07	13.79	14.7	6.19
0.8	12.74	13.75	7.34	12.08	13.05	7.43

The main contribution of the present paper is that the use of Gegenbauer polynomials as base functions in the expansion of stream function. The advantage of using Gegenbauer polynomials is that integration on double summation reduces to only single summation for drag calculation. This is due to the orthogonal nature of the Gegenbauer polynomials. Hence, we could obtain the simple expression for drag, equation (5.12), in comparison with the formula given by Liao [34].

Also, by the nature of orthogonality of Gegenbauer polynomials, RHS of (4.18) can be simplified more easily than in the case of Liao [33] and making it possible for us to go up to 13 iterations.

7. Conclusions

In this paper, the stream function ψ and vorticity function Ω for uniform flow past a sphere with slip boundary condition are found using HAM, using Gegenbauer polynomials. The expression for drag is derived. The flow pattern and coefficient of drag C_d are compared with the existing literature and are found in good agreement with [33], [34], [39], [42]. The solution by HAM is converging as σ is increasing in the auxiliary function $H=1/r^{\sigma}$. The values of C_d are obtained at $\sigma=1$ and $\sigma=2$ and found matching with Feng et al [42]. It is observed that

a) Analytical solution for stream lines and vorticity lines is obtained at moderately large Reynolds numbers $Re \leq 250$.
b) Drag coefficient C_d is in good agreement with error less than 14% with the results obtained by previous authors up to $Re \leq 50$ for small slip and large slip values ([39], [42]).
c) As Reynolds number increases, the size of wakes behind the sphere and the vorticity diffusion are increasing.
d) The calculations for drag and stream lines are done easily by the use of Gegenbauer polynomials in view of their orthogonality. Results up to 13^{th} order deformation are obtained.

Acknowledgements

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