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Fitted Upwind Difference Scheme for Solving Singularly Perturbed Differential- Difference Equations with Negative Shift

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Abstract: In this paper, a fitted upwind difference scheme has been presented for solving singularly perturbed differential-difference equations with negative shift. First, the singularly perturbed differential-difference equation has been replaced by an asymptotically equivalent singular perturbation problem. Then, a fitting factor is introduced into upwind finite difference scheme and obtained from the theory of singular perturbations. Then, a three term recurrence relation is obtained. The resulted tri diagonal system has been solved by Discrete Invariant Imbedding Algorithm. The efficiency of the method has been demonstrated by implementing on several model examples by taking different values for the delay and the perturbation parameters.

Keywords: Singular perturbation problems; differential-difference equations; delay parameter; boundary layer; perturbation parameter; finite differences.

1. Introduction

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing at least one delay term. Differential-difference equations, also known as delay differential equations arise frequently in mathematical modeling of various practical phenomenon in biosciences, engineering, ecology and control theory [2, 3], where the time evolution depends not only on present states but also on states at or near a given time in the past. Any system involving a feedback will almost always involve time delays. As a result these problems have received a lot of interest in recent times and many researchers have been trying to develop various methods for solving these problems. A delay differential equation is of retarded type if it does not involve delayed derivatives and it is said to be neutral type if it has delayed derivatives. Kadalbajoo and et al. [6, 7] constructed and analyzed a fitted operator finite difference method to solve problems arising from singularly perturbed differential difference equations. Reddy and et al. [12] presented a numerical integration method to solve delay differential equations. Variety of papers have been published in the recent years on singularly perturbed differential difference equations, some of them are Kadalbajoo and Reddy [6], Lange and Miura [8, 9], Pramod Chakravarthy and Rao [11].

In this paper, fitted upwind finite difference scheme is presented for solving a singularly perturbed delay differential equation with layer behavior. First, the singularly perturbed delay

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differential equation is replaced by an asymptotically equivalent second order singularly perturbed two point boundary value problem. Then, a fitting factor is introduced into upwind finite difference scheme and obtained from the theory of singular perturbations. Thomas algorithm is used to solve the obtained tri-diagonal system and the stability of the algorithm is also considered. To show the applicability of the method, we have applied it on several numerical examples by taking different values for perturbation and delay parameters.

2. Description of the Method

Consider the singularly perturbed differential-difference equation of the form:

$$\varepsilon y'' + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 < x < 1 \tag{1}$$

with boundary conditions

$$y(0) = \alpha, \text{ and } y(1) = \beta \tag{2}$$

Where $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions, $0<\varepsilon<<1$ and $\delta=o(\varepsilon)$ is the delay parameter such that $(\varepsilon - \delta a(x)) > 0$ for all. $x \in [0,1]$ Furthermore, α and β are positive constants. When δ is zero, equation (1) is reduced to a singularly perturbed ordinary differential equation with small ε which exhibits layer behavior and turning points depending upon the coefficient of convection term. The layer behavior of the problem under consideration is maintained for $\delta \neq 0$ but sufficiently small (i.e. $\delta = 0(\varepsilon)$) and in this paper we consider the problem where the layer behavior is maintained.

2.1. Left end boundary Layer problems

We assume that $a(x) \geq M > 0$ throughout the interval [0,1] for some positive constant M and $\varepsilon - \delta a(x) > 0, \forall x \in [0,1]$. This assumption implies that the boundary layer will be at the left end of the interval. That is in the neighborhood of $x = 0$.

Taking the Taylor series expansion of the term $y'(x-\delta)$ we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{3}$$

Substituting (3) into (1) we get an asymptotically equivalent two point boundary value pr oblem:

$$\varepsilon' y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{4}$$

with boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta \tag{5}$$

where $0 < \varepsilon' = \varepsilon - \delta\xi << 1$ such that $\xi = \min_{0 \leq x \leq 1} a(x)$.

The transformation from (1) to (4) is admitted because of the condition that δ is sufficiently small. For details of the validity of this transition see Els'golts & Norkin[4].

From the theory of singular perturbations, it is known that the solution of (4)-(5) is of the form [cf. O' Malley[10] pp.22-26]

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int_0^x \left(\frac{a(x)}{\varepsilon'} - \frac{b(x)}{a(x)}\right)dx} + o(\varepsilon') \tag{6}$$

where $y_0(x)$ is the solution of the reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \; y_0(1) = \beta$$

By taking Taylor series expansion for $a(x)$ and $b(x)$ about the point '0' and restricting to their first terms, (6) becomes

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon'} - \frac{b(0)}{a(0)}\right)x} + o(\varepsilon') \tag{7}$$

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih,\ i = 0,\ 1, 2\ldots N$.

From (7) we have $y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon'} - \frac{b(0)}{a(0)}\right)ih} + o(\varepsilon')$

Now taking the limit of both sides we get:

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0)-\varepsilon' b(0)}{a(0)}\right)i\rho} + o(\varepsilon') \tag{8}$$

where $\rho = h/\varepsilon'$.

Now we consider the second order upwind finite difference scheme in (4) and fitting a parameter σ we get:

$$\varepsilon'\sigma\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{y_{i+1} - y_i}{h}\right) + b(x_i)y(x_i) = f(x_i);\ \ 1 \le i \le N-1 \tag{9}$$

$y_0 = \alpha, y_N = \beta,$ where σ is a fitting factor which is to be determined in such a way that the solution of (9) converges uniformly to the solution of (1) - (2). Multiplying (9) by h and taking the limits as $h \to 0,$ we get:

$$\lim_{h\to 0}\left[\frac{\sigma}{\rho}\big(y(ih+h) - 2y(ih) + y(ih-h)\big) + a(ih)\big(y(ih+h) - y(ih)\big)\right] = 0 \tag{10}$$

Provided $f(x_i) - b(x_i)y_i$ is bounded. Substituting (8) in (10) and simplifying we get:

$$\lim_{h\to 0}\frac{\sigma}{\rho} = \frac{a(0)}{4}\left[\frac{(1 - e^{-\left(\frac{a^{2(0)}-\varepsilon' b(0)}{a(0)}\rho\right)}}{Sinh^2\left[\left(\frac{a^2(0) - \varepsilon' b(0)}{2a(0)}\right)\rho\right]}\right] \tag{11}$$

i.e., $$\sigma = \frac{\rho a(0)}{4}\left[\frac{(1 - e^{-\left(\frac{a^{2(0)}-\varepsilon' b(0)}{a(0)}\rho\right)}}{Sinh^2\left[\left(\frac{a^2(0) - \varepsilon' b(0)}{2a(0)}\right)\rho\right]}\right] \tag{12}$$

From (9) we have

$$\left(\frac{\sigma\varepsilon'}{h^2}\right)y_{i-1} - \left(\frac{2\sigma\varepsilon'}{h^2} + \frac{a_i}{h} - b_i\right)y_i + \left(\frac{\sigma\varepsilon'}{h^2} + \frac{a_i}{h}\right)y_{i+1} = f_i \tag{13}$$

where $a_i = a(x_i),\ b_i = b(x_i)$ and $f_i = f(x_i)$ considered for convenience. The above equation can be written as a three term recurrence relation:

$$E_i y_{i-1} - F_i y_i + G y_{i+1} = H_i,\ \ i = 1,\ 2,\ 3,\ \ldots,\ N-1 \tag{14}$$

Where

$$E_i = \frac{\sigma\varepsilon'}{h^2} \tag{15}$$

$$F_i = \frac{2\sigma\varepsilon'}{h^2} + \frac{a_i}{h} - b_i \tag{16}$$

$$G_i = \frac{\sigma\varepsilon'}{h^2} + \frac{a_i}{h} \tag{17}$$

$$H_i = f_i \tag{18}$$

This three term recurrence relation can be solved by Discrete Invariant Imbedding Algorithm described in the next section.

2.2. Thomas Algorithm

A brief discussion of the Thomas Algorithm which is also called Discrete-Invariant Imbedding Algorithm, Angel and Bellman [1] is presented as follows:
Consider the relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,\ 2, \ldots, N-1 \tag{19}$$

where E_i, F_i, G_i and H_i are known and

$$y_0 = y(0) = \alpha \tag{20}$$

$$y_N = y(1) = \beta \tag{21}$$

Consider a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \quad i = N-1, N-2, \ldots, 2, 1 \tag{22}$$

where W_i and T_i corresponding to $W(x_i)$ and $T(x_i)$ are to be determined. From (22) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{23}$$

Substituting (23) in (19), we get

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \tag{24}$$

By comparing (24) and (22) we get the recurrence relations

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{25}$$

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \tag{26}$$

To solve these recurrence relations for i = 1, 2, … $N-1$, we need to know the initial conditions for W_0 and T_0. This can be done by considering (20) which gives $y_0 = \alpha = W_0 y_1 + T_0$. If we choose W_0=0, then $T_0 = \alpha$. With these initial values, we compute sequentially W_i and T_i for i = 1, 2, … $N-1$ from (25) and (26) in the forward process and then obtain y_i in the backward process from (22) using (20).

For further discussion on the conditions for the Thomas Algorithm to be stable, one can refer (Angel and Bellman [1], Els'golts and Norkin [4] and Kadalbajoo and Reddy [5]). Here, under the assumptions that $a(x) > 0$, $b(x) < 0$ and $(\varepsilon - \delta a(x)) > 0$, the diagonal dominance property

$E_i + G_i = \frac{2\sigma\varepsilon'}{h^2} + \frac{a_i}{h} \le \frac{2\sigma\varepsilon'}{h^2} + \frac{a_i}{h} - b_i = F_i$, as $b_i < 0$ and $0 \le \frac{\sigma\varepsilon'}{h^2} = E_i \le \frac{\sigma\varepsilon'}{h^2} + \frac{a_i}{h} = G_i$. Hence, $F_i \ge E_i + G_i$ and $E_i > 0$, $G_i > 0$, $|E_i| \le |G_i|$ holds true and thus Thomas algorithm is stable.

2.3. Right End Boundary Layer Problems

Now we consider (4) - (5) and assume that $a(x) \le M < 0$ such that $0 < \varepsilon' = \varepsilon - \delta\zeta << 1$, where $\zeta = \max_{0\le x\le 1} a(x)$ throughout the interval [0, 1] and M is some negative constant. This assumption implies that the boundary layer will be at the right end of the interval, i.e. in the neighborhood of $x = 1$. Thus, from the theory of singular perturbations, it is known that the solution of (4) - (5) is of the form [cf.O' Malley[10]: 22-26]

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1\left(\frac{a(x)}{\varepsilon'} - \frac{b(x)}{a(x)}\right)dx} + o(\varepsilon') \tag{27}$$

where $y_0(x)$ is the solution of the reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x),\ y_0(0) = \alpha \tag{28}$$

By taking Taylor series expansion for $a(x)$ and $b(x)$ about the point '1' and restricting to their first terms, (27) becomes

$$y(x) = y_0(x) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon'} - \frac{b(1)}{a(1)}\right)(1-x)} + o(\varepsilon') \tag{29}$$

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih,\ i = 0, 1, 2,\ \ldots, N$. From (29) we have

$$y(ih) = y_0(ih) + (\beta - y_0(1))e^{\left(\frac{a(1)}{e'} - \frac{b(1)}{a(1)}\right)(1-ih)} + o(\varepsilon') \tag{30}$$

Therefore, taking the limit of both sides of (30) we get

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{\left(\frac{a^2(1) - \varepsilon' b(1)}{a(1)}\right)(\frac{1}{\varepsilon'} - i\rho)} + o(\varepsilon') \tag{31}$$

where $\rho = h/\varepsilon'$.

Now we consider the second order upwind finite difference scheme in (4)

$$\varepsilon'\sigma\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{y_i - y_{i-1}}{h}\right) + b(x_i)y(x_i) = f(x_i);\ 1 \le i \le N-1 \tag{32}$$

$y_0 = \alpha,\ y_N = \beta,$ where σ is a fitting factor which is to be determined in such a way that the solution of (32) converges uniformly to the solution of (1) - (2). Multiplying (32) by h and taking the limits as $h \to 0,$ we get

$$\lim_{h\to 0}\left[\frac{\sigma}{\rho}\big(y(ih+h) - 2y(ih) + y(ih-h)\big) + a(ih)\big(y(ih) - y(ih-h)\big)\right] = 0 \tag{33}$$

provided $f(x_i) - b(x_i)y_i$ is bounded.

Substituting (31) in (33) and simplifying, we get the fitting factor as:

$$\lim_{h\to 0}\frac{\sigma}{\rho}=\frac{a(0)}{4}\left[\frac{(e^{\left(\frac{a^2(1)-\varepsilon' b(1)}{a(1)}\rho\right)}-1)}{Sinh^2\left(\left(\frac{a^2(1)-\varepsilon' b(1)}{a(1)}\right)\frac{\rho}{2}\right)}\right] \tag{34}$$

i.e., $$\sigma=\frac{\rho a(0)}{4}\left[\frac{(e^{\left(\frac{a^2(1)-\varepsilon' b(1)}{a(1)}\rho\right)}-1)}{Sinh^2\left(\left(\frac{a^2(1)-\varepsilon' b(1)}{a(1)}\right)\frac{\rho}{2}\right)}\right] \tag{35}$$

From (32) we have

$$\left(\frac{\sigma\varepsilon'}{h^2}-\frac{a_i}{h}\right)y_{i-1}-\left(\frac{2\sigma\varepsilon'}{h^2}-\frac{a_i}{h}-b_i\right)y_i+\left(\frac{\sigma\varepsilon'}{h^2}\right)y_{i+1}=f_i \tag{36}$$

where $a_i=a(x_i),\ b_i=b(x_i)$ and $f_i=f(x_i)$ are considered for convenience. Thus, equation (36) can be written as the three term recurrence relation of the form

$$E_i y_{i-1}-F_i y_i+G y_{i+1}=H_i,\quad i=1,2,3,\ \ldots,\ N-1 \tag{37}$$

Where

$$E_i=\frac{\sigma\varepsilon'}{h^2}-\frac{a_i}{h} \tag{38}$$

$$F_i=\frac{2\sigma\varepsilon'}{h^2}-\frac{a_i}{h}-b_i \tag{39}$$

$$G_i=\frac{\sigma\varepsilon'}{h^2} \tag{40}$$

$$H_i=f_i \tag{41}$$

This three term recurrence relation can be easily solved by Discrete-Invariant Imbedding Algorithm described in section 2.2.

3. Numerical examples

To illustrate the applicability of the method, some numerical examples with left-end and right-end boundary layer are considered. The computed results are compared with exact solution for problems whose exact solution is known and for the problems whose exact solutions are not known, solution is calculated for different values of ε and δ.

Example 1.

Consider the singularly perturbed differential difference equation with left end boundary layer $\varepsilon y''(x)+y'(x-\delta)-y(x)=0;\quad x\in[0,\ 1]$ with $y(0)=1$ and $y(1)=1$.

The exact solution to this problem is given by

$$y(x)=\frac{\left(1-e^{m_2}\right)e^{m_1x}+\left(e^{m_1}-1\right)e^{m_2x}}{e^{m_1}-e^{m_2}}$$

where $m_1 = \dfrac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$ and $m_2 = \dfrac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$

Computational results are presented in the tables 1, 2, 3 and 4 for $\varepsilon = 0.01$ and 0.001 for different values of δ. The effect of δ on the boundary layer is shown in graph (Figure 1) for different values of δ.

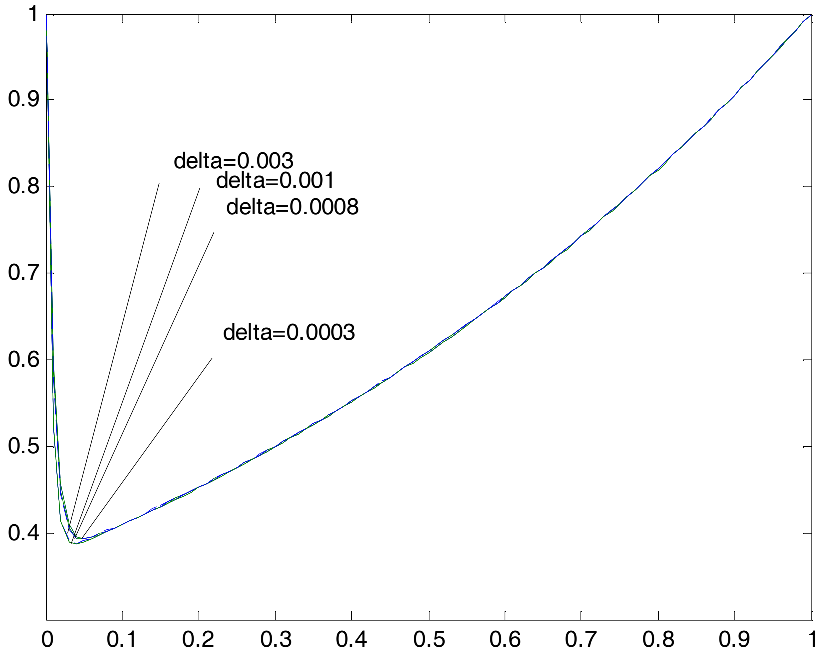

Figure 1. Graph of numerical solution of Example 1 for different values of δ.

Table 1. Numerical results of Example 1 for $\varepsilon = 0.01$, $\delta = 0.001$, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.04	0.3932775	0.3932546	2.291e-05
0.05	0.3925070	0.3923166	1.903e-04
0.06	0.3949016	0.3946419	2.597e-04
0.07	0.3983441	0.3980571	2.869e-04
0.09	0.4061028	0.4058020	3.008e-04
0.20	0.4528155	0.4525184	2.971e-04
0.40	0.5520024	0.5517307	2.717e-04
0.60	0.6729157	0.6726949	2.208e-04
0.80	0.8203144	0.8201798	1.346e-04
1.00	1.0000000	1.0000000	0.000e+00

Table 2. Numerical results of Example 1 for $\varepsilon = 0.01$, $\delta = 0.003$, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.03	0.3901927	0.3900410	1.516e-04
0.04	0.3877603	0.3874347	3.256e-04
0.05	0.3901408	0.3897593	3.813e-04
0.07	0.3975241	0.3971217	4.024e-04
0.09	0.4054658	0.4050624	4.033e-04
0.30	0.4993732	0.4989908	3.823e-04
0.50	0.6089609	0.6086279	3.330e-04
0.70	0.7425978	0.7423541	2.437e-04
0.90	0.9055613	0.9054623	9.907e-05
1.00	1.0000000	1.0000000	0.000e+00

Table 3. Numerical results of Example 1 for $\varepsilon = 0.001$, $\delta = 0.0003$, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.02	0.3771424	0.3755682	1.574e-03
0.03	0.3809138	0.3793401	1.573e-03
0.05	0.3885702	0.3869979	1.572e-03
0.07	0.3963804	0.3948102	1.570e-03
0.08	0.4003442	0.3987754	1.568e-03
0.20	0.4511179	0.4495803	1.537e-03
0.40	0.5504496	0.5490418	1.407e-03
0.60	0.6716531	0.6705074	1.145e-03
0.80	0.8195444	0.8188452	6.992e-04
1.00	1.0000000	1.0000000	0.000e+00

Table 4. Numerical results of Example 1 for $\varepsilon = 0.001$, $\delta = 0.0008$ N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.02	0.3771424	0.3753846	1.757e-03
0.03	0.3809138	0.3791565	1.757e-03
0.04	0.3847229	0.3829663	1.756e-03
0.06	0.3924559	0.3907012	1.754e-03
0.08	0.4003442	0.3985923	1.751e-03
0.20	0.4511179	0.4494008	1.717e-03
0.40	0.5504496	0.5488774	1.572e-03
0.60	0.6716531	0.6703736	1.279e-03
0.80	0.8195444	0.8187634	7.809e-04
1.00	1.0000000	1.0000000	0.000e+00

Example 2.

Consider an example with variable coefficient singularly perturbed differential difference equation with left layer
$\varepsilon y''(x)+e^{-0.5x}y'(x-\delta)-y(x)=0$ with $y(0)=1,\quad y(1)=1$.

The exact solution is not known for this problem. The computational results obtained for different values ε and δ are presented in Tables 5 and 6. Further, the effect of δ on the boundary layer is shown in graph (Figure 2) for different values of δ.

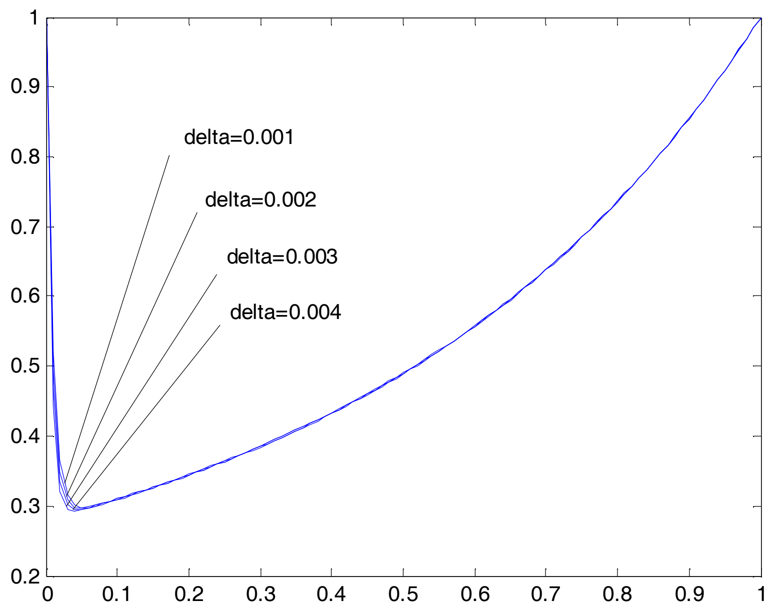

Figure 2. Graph of numerical solution of Example 2 for different values of δ.

Table 5. Numerical results for Example 2 with $\varepsilon=0.01$, $N=100$, different values of δ

x	$\delta=0.00$	δ =0.001	δ =0.002	δ =0.003	δ =0.004
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3815079	0.3651163	0.3492111	0.3341466	0.3203373
0.04	0.3052303	0.3008469	0.2972621	0.2944642	0.2923952
0.06	0.3002676	0.2990323	0.2980527	0.2972743	0.2966421
0.08	0.3049404	0.3042990	0.3037208	0.3031871	0.3026848
0.10	0.3110740	0.3105300	0.3100047	0.3094949	0.3090002
0.20	0.3458089	0.3452682	0.3447352	0.3442122	0.3437016
0.40	0.4344148	0.4338594	0.4333108	0.4327711	0.4322427
0.60	0.5586200	0.5580990	0.5575832	0.5570742	0.5565743
0.80	0.7369857	0.7366082	0.7362334	0.7358624	0.7354968
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6. Numerical results of Example 2 for $\varepsilon = 0.001$, *N*=100, different values of δ

x	$\delta = 0.0001$	δ =0.0002	δ =0.0003	δ =0.0004	δ =0.0008
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.2792298	0.2792209	0.2792164	0.2792144	0.2792134
0.03	0.2848683	0.2848681	0.2848681	0.2848681	0.2848681
0.05	0.2906957	0.2906956	0.2906956	0.2906955	0.2906955
0.07	0.2967025	0.2967024	0.2967023	0.2967023	0.2967023
0.09	0.3028953	0.3028952	0.3028951	0.3028951	0.3028951
0.10	0.3060636	0.3060635	0.3060635	0.3060634	0.3060634
0.20	0.3406138	0.3406137	0.3406136	0.3406136	0.3406136
0.40	0.4289315	0.4289313	0.4289313	0.4289312	0.4289312
0.60	0.5533209	0.5533208	0.5533207	0.5533207	0.5533207
0.80	0.7330194	0.7330193	0.7330193	0.7330193	0.7330193
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Example 3.

Consider the singularly perturbed differential difference equation with right-end boundary layer
$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0; x \in [0, 1]$ with $y(0) = 1$ and $y(1) = -1$.

The exact solution is given by $y(x) = \dfrac{\left(1+e^{m_2}\right)e^{m_1 x} - \left(e^{m_1}+1\right)e^{m_2 x}}{e^{m_2} - e^{m_1}}$

where $m_1 = \dfrac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$ and $m_2 = \dfrac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$

Computational results are presented in the Tables 7, 8 and 9 for $\varepsilon = 0.01$ and 0.001 for different values of δ. The effect of δ on the boundary layer is shown in graph (Figure 3) for different values of δ.

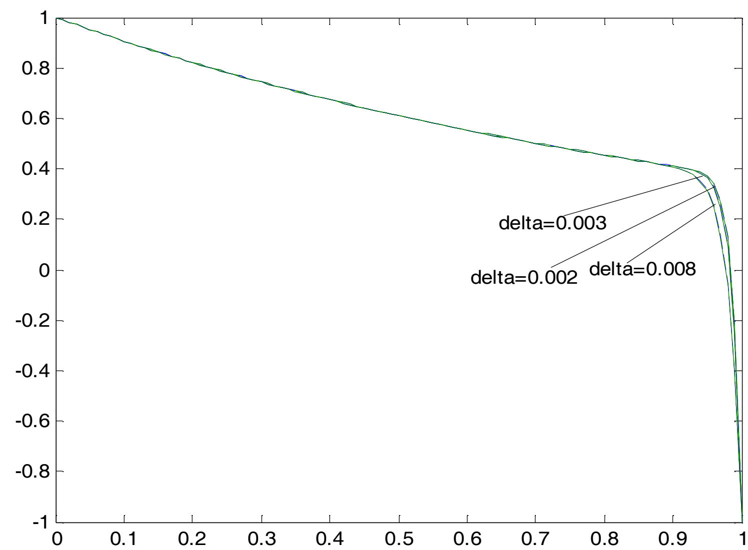

Figure 3. Graph of numerical solution of Example 3 for different values of δ.

Table 7. Numerical results Example 3 for $\varepsilon = 0.01, \delta = 0.002$, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000
0.20	0.8207411	0.8206521	8.898e-05
0.40	0.6736160	0.6734699	1.460e-04
0.60	0.5528644	0.5526845	1.798e04
0.80	0.4537585	0.4535617	1.967e-04
0.90	0.4108110	0.4105821	2.289e-04
0.91	0.4064063	0.4061459	2.604e-04
0.93	0.3955821	0.3951281	4.540e-04
0.95	0.3720079	0.3708222	1.185e-04
0.97	0.2774855	0.2740694	3.416e-03
1.00	-1.0000000	-1.0000000	0.000e+00

Table 8. Numerical results for Example 3, $\varepsilon = 0.01$, $\delta = 0.003$, N=100

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000
0.20	0.8208852	0.8208084	7.678e-05
0.40	0.6738526	0.6737265	1.260e-04
0.60	0.5531556	0.5530004	1.552e-04
0.80	0.4540771	0.4539072	1.699e-04
0.91	0.4062358	0.4059562	2.796e-04
0.93	0.3939262	0.3933562	5.700e-04
0.95	0.3650479	0.3635169	1.531e-04
0.97	0.2552733	0.2511992	4.074e-04
0.98	0.0969142	0.0910572	5.857e-04
1.00	-1.0000000	-1.0000000	0.000e+00

Table 9. Numerical results for Example 3 for $\varepsilon = 0.001$, $\delta = 0.008$

x	Numerical solution	Exact solution	Absolute Error
0.00	1.0000000	1.0000000	0.000
0.20	0.8216071	0.8215815	2.559e-05
0.40	0.6750383	0.6749962	4.204e-05
0.60	0.5546163	0.5545645	5.182e-05
0.80	0.4556629	0.4556031	5.982e-05
0.91	0.4012516	0.4004645	7.870e-04
0.93	0.3765548	0.3747227	1.832e-03
0.95	0.3158472	0.3118252	4.021e-03
0.96	0.2518080	0.2461418	5.666e-03
0.97	0.1409865	0.1334910	7.495e-03
1.00	-1.0000000	-1.0000000	0.000

Example 4.

Consider an example with variable coefficient singularly perturbed differential difference equation with right layer:

$\varepsilon y''(x) - e^{x} y'(x-\delta) - y(x) = 0$ with $y(0) = 1, \quad y(1) = 1$

The exact solution is not known for this problem. The computational results obtained for different values ε and δ are presented in Tables 10 and 11. Further, the effect of δ on the boundary layer is plotted in graph (Figure 4) for different values of δ.

Table 10. Numerical results for Example 4, $\varepsilon = 0.01, \quad N = 100,$ different values of δ

x	$\delta = 0.00$	$\delta = 0.003$	$\delta = 0.006$	$\delta = 0.008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8370671	0.8380376	0.8388649	0.8393793
0.40	0.7233046	0.7247353	0.7259784	0.7267571
0.60	0.6415895	0.6432350	0.6446872	0.6456024
0.80	0.5815045	0.5832440	0.5847992	0.5857843
0.90	0.5574139	0.5591836	0.5608834	0.5621501
0.91	0.5551863	0.5569733	0.5588151	0.5603198
0.93	0.5508256	0.5527974	0.5556074	0.5583636
0.95	0.5466615	0.5501960	0.5572984	0.5639977
0.97	0.5450029	0.5608573	0.5840460	0.6007747
0.99	0.6196261	0.6921538	0.7421622	0.7673373
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 11. Numerical results for Example 4, $\varepsilon = 0.001, \quad N = 100,$ different values of δ

x	$\delta = 0.0001$	$\delta = 0.0003$	$\delta = 0.0006$	$\delta = 0.0008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8356442	0.8356508	0.8356796	0.8357099
0.40	0.7213164	0.7213259	0.7213675	0.7214117
0.60	0.6394073	0.6394179	0.6394649	0.6395154
0.80	0.5792885	0.5792995	0.5793486	0.5794017
0.90	0.5552094	0.5552203	0.5552698	0.5553234
0.91	0.5529835	0.5529944	0.5530439	0.5530976
0.93	0.5486239	0.5486348	0.5486843	0.5487380
0.95	0.5443838	0.5443947	0.5444442	0.5444980
0.97	0.5402593	0.5402703	0.5403200	0.5403755
0.99	0.5363122	0.5369482	0.5401521	0.5438243
1.00	1.0000000	1.0000000	1.0000000	1.0000000

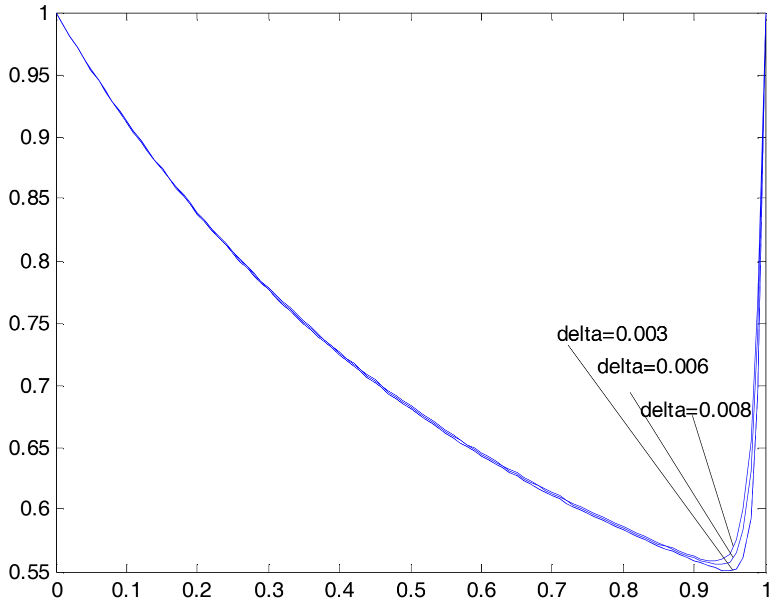

Figure 4. Graph of numerical solution of Example 4 for different values of δ.

5. Discussion and conclusion

A fitted upwind difference scheme has been presented for solving singularly perturbed differential-difference equations whose solutions exhibits boundary layer behavior. The scheme is to be repeated for different choices of the delay parameter δ and perturbation parameter ε. The choice of δ is not unique, but can assume any number of values satisfying the condition, $0<\delta<<1$ and $0<\varepsilon'<<1$ such that $\varepsilon'=\varepsilon-\delta\alpha(x),\ \forall x\in[0,\ 1]$. The solutions are calculated for all the values of h but only few values have been reported. It can be observed from the tables that the proposed method approximates the exact solution very well (Tables 1-4, 7-9) and also produces good and/or consistent results which are in support of the theory for different values of ε and δ for problems without exact solutions (Tables 5-6, 10-11) which in turn implies the efficiency of the method. The delay parameter δ affects both the boundary layer solutions (left and right) in similar fashion but reversely. That is as δ increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases (Figures 1-4).

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