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A Fitted Arithmetic Average Three–Point Finite Difference Method for Singularly Perturbed Two–Point Boundary Value Problems with Dual layers

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Abstract: In this paper, an exponentially fitted arithmetic average difference scheme is proposed to solve singularly perturbed differential equations with dual layer behaviour. In this method, we have extended the arithmetic average finite difference method to the second order singularly perturbed two-point boundary value problem. We have introduced a fitting factor in a three point arithmetic average discretization for the given problem which takes care of the rapid changes that occur in the boundary layers due to the perturbation parameter. This fitting factor is obtained from the asymptotic approximate solution of singular perturbations. The discrete invariant imbedding algorithm is used to solve the tridiagonal system of the fitted method. Maximum absolute errors of the several numerical examples are presented to illustrate the proposed method.

Keywords: Singular perturbation problems; dual layer; three-point arithmetic average discretization; boundary layer; fitting factor

1. Introduction

Singular perturbation problem now is a maturing mathematical area with long history and a strong promise for continued important applications throughout science and engineering. Singular perturbation problems arise in various fields of engineering and applied sciences such as fluid dynamics, electrical networks, and many other areas. Typical examples of Singular Perturbation Problems include Navier-Stokes equation of fluid at high Reynolds number, heat transport problem with Peclet numbers, magneto-hydrodynamics duct problems with Hartman number. A differential equation with a small positive parameter multiplying the highest derivative term is generally called the Singular Perturbation Problem.

For a detailed theoretical and analytical discussion on this topic, one may refer to the references [1-8]. The survey papers by Kadalbajoo and Reddy [9], Kadalbajoo and Patidar [10] give an erudite outline of the singular perturbation problems and their treatment on fluid dynamical boundary layers. A set of general sufficient conditions for a uniformly convergent scheme for singularly perturbed turning point problem is obtained by Farrell [11]. Natesan and Ramanujam [12] derived a computational method for the singularly perturbed turning point

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problem in which exponentially fitted difference schemes are combined with classical numerical methods. Natesan et. al. [13] proposed a parameter uniform numerical method on Shishkin mesh to solve singularly perturbed turning point problems. Miller et al. [14] elucidate the classical schemes on Shishkin meshes to solve singularly perturbed BVPs of convection–diffusion and reaction–diffusion problems subject to Dirichlet boundary conditions. Phaneendra et. al. [15] proposed a fitting factor in Numerov method to solve singular perturbation problems with twin boundary layers. The solution of singular perturbation problem exhibits boundary layers. A boundary layer is a narrow region in which solution of the problem changes rapidly. For these problems, the existing numerical methods produce good results only if we take $h << \varepsilon$, where h is mesh size and ε is the perturbation parameter. But this is costly and time consuming process. If we take $h \geq \varepsilon$, the existing numerical methods produce oscillatory solution and pollute the solution in the entire interval, because of the boundary layer behavior. Thus, in this paper we proposed an efficient and simpler computational technique to solve singularly perturbed two-point boundary value problems. In this paper, we proposed an exponentially fitted arithmetic average difference scheme on a uniform mesh for solving singularly perturbed two-point boundary value problems exhibiting dual boundary layers. In section 2, we described the fitted arithmetic finite difference method by extending the arithmetic finite difference scheme to the second order singularly perturbed two-point boundary value problem. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems and the results are given in Section 3. Finally the discussions and conclusion are given in the last section.

2. Numerical method

To describe the method, we consider the singularly perturbed two point boundary value problem of the form:

$$\varepsilon y'' - b(x)y(x) = f(x); \qquad x \in (0,1), \tag{1}$$

with boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ (2)

Since, the problem (1) exhibits dual (twin) layers, we consider the asymptotic expansion solution of for the problem (1) and (2) (Doolan et. al. [16]):

$$y(x,\varepsilon) = \sum_{i=0}^{\infty} \left[y_i(x) + v_i(\tau) + w_i(\eta) \right] \varepsilon^i \quad , \tag{3}$$

where $\tau = x/\sqrt{\varepsilon}$ and $\eta = (1-x)/\sqrt{\varepsilon}$. Then the zeroth order of the above asymptotic expansion is given by

$$y(x) = y_0(x) + v_0(\tau) + w_0(\eta) \tag{4}$$

where

$$y_0(x) = \frac{f(x)}{b(x)} \tag{5}$$

is the solution of the reduced problem of (1) and (2), which does not satisfy both the boundary conditions, v_0 is the left boundary layer correction and w_0 is the right boundary layer correction.

The solutions v_0, w_0 satisfy the differential equations

$$\frac{d^2 v_0(\tau)}{d\tau^2} - b(0) v_0(\tau) = 0; \qquad \tau \in (0, \infty) \tag{6}$$

$$\frac{d^2 w_0(\eta)}{d\eta^2} - b(1) w_0(\eta) = 0; \qquad \eta \in (0, \infty) \tag{7}$$

with $v_0(\tau = 0) + w_0(\eta = 1/\sqrt{\varepsilon}) = \alpha - y_0(0)$

$$v_0(\tau = 1/\sqrt{\varepsilon}) + w_0(\eta = 0) = \beta - y_0(1)$$

$$v_0(\tau = \infty) = w_0(\eta = \infty) = 0$$

Solutions of (6) and (7) are given by

$$v_0(\tau) = Ae^{-\sqrt{b(0)}\tau} \tag{8}$$

$$w_0(\eta) = Be^{-\sqrt{b(1)}\eta} \tag{9}$$

Therefore, zeroth order solution of (1) and (2) becomes

$$y(x) = y_0(x) + Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x} + Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x)}$$

where *A* and *B* are given by

$$A = \frac{\left(\beta - y_0(1)\right) - \left(\alpha - y_0(0)\right)e^{-\sqrt{\frac{b(0)}{\varepsilon}}}}{1 - e^{-\frac{\left(\sqrt{b(0)}+\sqrt{b(1)}\right)}{\sqrt{\varepsilon}}}} \tag{10}$$

$$B = \frac{\left(\alpha - y_0(0)\right) - \left(\beta - y_0(1)\right)e^{-\sqrt{\frac{b(1)}{\varepsilon}}}}{1 - e^{-\frac{\left(\sqrt{b(0)}+\sqrt{b(1)}\right)}{\sqrt{\varepsilon}}}} \tag{11}$$

We rewrite the differential equation $\varepsilon y'' - b(x)y(x) = f(x)$ as

$\varepsilon y''(x) = g(x, y)$ where $g(x, y) = b(x)y(x) + f(x)$.

Now we divide the interval [0, 1] into *N* equal parts with constant mesh length *h*. Let $0 = x_0, x_1, x_2, ..., x_N = 1$ be the mesh points. Then we have $x_i = ih; i = 0,1,..., N$. We choose *n* such that $x_n = \frac{1}{2}$. Then in $\left[0, \frac{1}{2}\right]$ the boundary layer will be in the left hand side i.e., at $x = 0$ and in $\left[\frac{1}{2}, 1\right]$ the boundary layer will be in the right hand side i.e., at $x = 1$.

At $x = x_i$ the above differential equation can be written as

$$\varepsilon y_i''(x) = g(x_i, y_i) \text{ where } g(x_i, y_i) = b(x_i)y(x_i) + f(x_i) \tag{12}$$

We consider the three-point arithmetic average discretization given by Chawla [17] for the problem (1):

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)=\frac{1}{3}(g_{i-1/2}+g_i+g_{i+1/2})$$

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)=\frac{1}{3}\left(b_{i-1/2}y_{i-1/2}+f_{i-1/2}+b_iy_i+f_i+b_{i+1/2}y_{i+1/2}+f_{i+1/2}\right) \tag{13}$$

Now, let $y_{i+1/2}=\frac{1}{2}(y_{i+1}+y_i)$ and $y_{i-1/2}=\frac{1}{2}(y_i+y_{i-1})$

Then equation (13), becomes

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{6}\left(b_{i-1/2}y_{i-1}+\left(b_{i-1/2}+2b_i+b_{i+1/2}\right)y_i+b_{i+1/2}y_{i+1}\right)=\frac{1}{3}\left(f_{i-1/2}+f_i+f_{i+1/2}\right) \tag{14}$$

In the interval $\left[0,\frac{1}{2}\right]$, we introduce a fitting factor σ_1 in the difference scheme (14) as

$$\sigma_1\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{6}\left(b_{i-1/2}y_{i-1}+\left(b_{i-1/2}+2b_i+b_{i+1/2}\right)y_i+b_{i+1/2}y_{i+1}\right)=\frac{1}{3}\left(f_{i-1/2}+f_i+f_{i+1/2}\right) \tag{15}$$

for $i=1,2,...,n-1$. To find the value of σ_1 on the left boundary layer, we use the left boundary layer asymptotic solution

$$v_0(x_i)=y_i=Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i} \tag{16}$$

and A is given by (10). We assume that solution converges uniformly to the solution of (1), then $f_{i-1/2}+f_i+f_{i+1/2}$ is bounded. As $h\to 0$, equation (15) becomes

$$\lim_{h\to 0}\frac{\sigma_1}{\rho^2}\left(y_{i-1}-2y_i+y_{i+1}\right)=\frac{b(0)}{6}\lim_{h\to 0}\left(y_{i-1}+4y_i+y_{i+1}\right) \tag{17}$$

where $\rho=\frac{h}{\sqrt{\varepsilon}}$. Using (16), we get

$$y_{i-1}=Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i}e^{\sqrt{b(0)}\rho} \quad \text{and} \quad y_{i+1}=Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i}e^{-\sqrt{b(0)}\rho}$$

Then, we get

$$y_{i-1}-2y_i+y_{i+1}=Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i}\left(e^{\sqrt{b(0)}\rho}-2+e^{-\sqrt{b(0)}\rho}\right)$$

$$y_{i-1}+4y_i+y_{i+1}=Ae^{-\sqrt{\frac{b(0)}{\varepsilon}}x_i}\left(e^{\sqrt{b(0)}\rho}+4+e^{-\sqrt{b(0)}\rho}\right)$$

Substituting these expressions in (17) and simplifying, we get the fitting factor

$$\sigma_1=\frac{\rho^2 b(0)\left(e^{\sqrt{b(0)}\rho}+e^{-\sqrt{b(0)}\rho}+4\right)}{24Sinh^2\left(\frac{\sqrt{b(0)}\rho}{2}\right)} \tag{18}$$

This will be the fitting factor in the interval $\left[0,\frac{1}{2}\right]$. Substituting the fitting factor (18) in (15), we have the three term recurrence relation as

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i \quad \text{for } i=1,2,...,n-1. \tag{19}$$

where

$$E_i=\frac{\varepsilon\sigma_1}{h^2}-\frac{b_{i-1/2}}{6},\; F_i=\left(\frac{2\varepsilon\sigma_1}{h^2}+\frac{1}{6}\left(b_{i-1/2}+2b_i+b_{i+1/2}\right)\right),\; G_i=\frac{\varepsilon\sigma_1}{h^2}-\frac{b_{i+1/2}}{6}$$

and $H_i=\frac{1}{3}\left(f_{i-1/2}+f_i+f_{i+1/2}\right)$

In the interval $\left[\frac{1}{2},1\right]$, the boundary layer will be in the right hand side i.e., at $x=1$. We introduce a fitting factor σ_2 in the difference scheme (15) as

$$\sigma_2\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right)-\frac{1}{6}\left(b_{i-1/2}y_{i-1}+\left(b_{i-1/2}+2b_i+b_{i+1/2}\right)y_i+b_{i+1/2}y_{i+1}\right)=\frac{1}{3}\left(f_{i-1/2}+f_i+f_{i+1/2}\right) \tag{20}$$

for $i = n+1, n+2, ...N-1$. To find the value of σ_2 on the right boundary layer, we use the right boundary layer asymptotic solution

$$w_0(x_i)=y_i=Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x_i)} \tag{21}$$

where B is given by (11). Assume that solution converges uniformly to the solution of (1), then $f_{i-1/2}+f_i+f_{i+1/2}$ is bounded. As $h\to 0$, equation (20) becomes

$$\lim_{h\to 0}\frac{\sigma_1}{\rho^2}\left(y_{i-1}-2y_i+y_{i+1}\right)=\frac{b(0)}{6}\lim_{h\to 0}\left(y_{i-1}+4y_i+y_{i+1}\right) \tag{22}$$

where $\rho=\frac{h}{\sqrt{\varepsilon}}$. Using (21), we get

$y_{i-1} = Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x_i)} e^{-\sqrt{b(1)}\rho}$ and $y_{i+1} = Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x_i)} e^{\sqrt{b(1)}\rho}$

Then, we get

$$y_{i-1} - 2y_i + y_{i+1} = Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x_i)} \left(e^{-\sqrt{b(1)}\rho} - 2 + e^{\sqrt{b(1)}\rho}\right)$$

$$y_{i-1} + 4y_i + y_{i+1} = Be^{-\sqrt{\frac{b(1)}{\varepsilon}}(1-x_i)} \left(e^{-\sqrt{b(1)}\rho} + 4 + e^{\sqrt{b(1)}\rho}\right)$$

Substituting these in (22) and simplifying, we get the fitting factor as

$$\sigma_2 = \frac{\rho^2 b(1)\left(e^{\sqrt{b(1)}\rho} + e^{-\sqrt{b(1)}\rho} + 4\right)}{24Sinh^2\left(\frac{\sqrt{b(1)}\rho}{2}\right)} \quad (23)$$

This will be the fitting factor in the interval $\left[\frac{1}{2}, 1\right]$. Now from the equation (20), we have the three term recurrence relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad \text{for } i = n+1, n+2, ..., N-1. \quad (24)$$

where

$$E_i = \frac{\varepsilon\sigma_2}{h^2} - \frac{b_{i-1/2}}{6},\ F_i = \left(\frac{2\varepsilon\sigma_2}{h^2} + \frac{1}{6}\left(b_{i-1/2} + 2b_i + b_{i+1/2}\right)\right),\ G_i = \frac{\varepsilon\sigma_2}{h^2} - \frac{b_{i+1/2}}{6}$$

and $H_i = \frac{1}{3}\left(f_{i-1/2} + f_i + f_{i+1/2}\right)$

We solve the above tridiagonal system (19), (24) by discrete invariant imbedding algorithm together with the value of $y_n = y\left(x = \frac{1}{2}\right)$ which is obtained by the solution of the reduced problem, i.e., $y_0(x)$.

3. Numerical Examples

To demonstrate the applicability of the method, we have applied it to four linear singular perturbation problems with dual boundary layers. These examples have been chosen because they have been widely discussed in literature and because exact solutions are available for comparison. Maximum absolute errors are presented in tables. We calculate the maximum absolute errors in the solution by the principle $G_\varepsilon^N = \max_{x_i \in D_\varepsilon^N} \left|Y_e^N(x_i) - Y^N(x_i)\right|$. Here $Y_e^N(x_i)$ denotes the exact solution at $x = x_i$ and $Y^N(x_i)$ denotes the numerical solution at $x = x_i$.

Example 1.
Consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) - y(x) = \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x\,;\ x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 0$.

The exact solution is given by

$$y(x) = \frac{\left(e^{\left(-(1-x)/\sqrt{\varepsilon}\right)} + e^{\left(-x/\sqrt{\varepsilon}\right)}\right)}{\left(1 + e^{-1/\sqrt{\varepsilon}}\right)} - \cos^2 \pi x$$

The maximum absolute errors are presented for different values of h and ε in Table 1 with fitting factor and without fitting factor.

Table 1. The maximum errors in solution of Example 1 for small values of $\varepsilon \le h$

ε \ h	2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f
10^{-3}	1.23(-2)	1.71(-1)	1.10(-3)	6.76(-2)	7.63(-5)	1.74(-2)	4.90(-6)	4.10(-3)
10^{-4}	2.35(-2)	2.75(-1)	4.60(-3)	2.13(-1)	5.60(-4)	1.19(-1)	4.47(-5)	4.18(-2)
10^{-5}	2.52(-2)	2.91(-1)	6.20(-3)	2.68(-1)	1.40(-3)	2.42(-1)	2.39(-4)	1.80(-1)

*with f.f = with fitted arithmetic average finite difference scheme
*without f.f.= arithmetic average finite difference scheme (Chawla and Shivakumar [17])

Example 2.
Consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) - y(x) = -1\,;\ x \in [0,1]$$

with y(0) = 0 and y(1) = 0.

The exact solution is $y(x) = 1 - e^{-x/\sqrt{\varepsilon}} - e^{-(1-x)/\sqrt{\varepsilon}}$. The maximum absolute errors are presented for different values of h and ε in Table 2 with and without fitting factor.

Table 2. The maximum errors in solution of Example 2 for small values of $\varepsilon \le h$

ε \ h	2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f
10^{-3}	1.84(-14)	1.50(-1)	1.84(-14)	6.23(-2)	1.84(-14)	1.64(-2)	1.84(-14)	3.80(-3)
10^{-4}	2.22(-16)	2.66(-1)	2.22(-16)	2.60(-1)	2.22(-16)	2.40(-1)	2.22(-16)	1.80(-1)
10^{-5}	2.22(-16)	2.66(-1)	2.22(-16)	2.60(-1)	2.22(-16)	2.40(-1)	2.22(-16)	1.80(-1)

Example 3.
Consider the following variable coefficient singular perturbation problem

$$\varepsilon y''(x) - (2 - x^2) y(x) = -1; \qquad x \in [-1, 1]$$

with y(-1) = 0 and y(1) = 0.

The exact solution is $y(x) = \frac{1}{2 - x^2} - e^{\frac{-(1+x)}{\sqrt{\varepsilon}}} - e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}$. The maximum absolute errors are presented for different values of *h* and ε in Table 3 with and without fitting factor.

Table 3. The maximum errors in solution of Example 3 for small values of $\varepsilon \le h$

h / ε	2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f
10^{-3}	2.46(-2)	1.49(-1)	2.09(-2)	7.91(-2)	1.74(-2)	3.20(-2)	1.72(-2)	2.05(-2)
10^{-4}	2.16(-2)	2.22(-1)	1.14(-2)	1.98(-1)	7.10(-3)	1.20(-1)	5.60(-3)	4.66(-2)
10^{-5}	2.16(-2)	2.33(-1)	1.12(-2)	2.45(-1)	5.60(-3)	2.34(-1)	2.80(-3)	1.79(-1)

Example 4.
Consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) - y(x) = 1 + 2\sqrt{\varepsilon}\left[e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}} \right]; \; x \in [0,1]$$

with y(0) = 0 and y(1) = 0.

The exact solution is given by $y(x) = 1 - (1-x)e^{-x/\sqrt{\varepsilon}} - xe^{(x-1)/\sqrt{\varepsilon}}$. The maximum absolute errors are presented for different values of *h* and ε with and without fitting factor in Table 4.

Table 4. The maximum errors in solution of Example 4 for small values of $\varepsilon \le h$

h / ε	2^{-3}		2^{-4}		2^{-5}		2^{-6}	
	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f	with f.f	without f.f
10^{-3}	8.54(-4)	1.52(-1)	3.10(-4)	6.39(-2)	3.19(-5)	1.70(-2)	2.12(-6)	4.00(-3)
10^{-4}	1.24(-5)	2.50(-1)	1.84(-4)	2.07(-1)	2.31(-4)	1.18(-1)	5.03(-5)	4.19(-2)
10^{-5}	5.50(-12)	2.66(-1)	1.07(-7)	2.60(-1)	1.35(-5)	2.40(-1)	8.01(-5)	1.80(-1)

4. Discussions and conclusion

We proposed an exponentially fitted arithmetic average finite difference method for solving singularly perturbed two-point boundary value problems with boundary layer at both (left and right) end points. To take care of the rapid changes that occur in the boundary layers due to small perturbation parameter, we introduced a fitting factor in three – point arithmetic average finite difference scheme. We obtained the value of the fitting factor from the asymptotic approximate solution of singular perturbations. We have implemented the present method on standard test problems. Maximum absolute errors of the numerical experiments are presented in tables. We compare the results by the present fitted scheme with the results by the three – point arithmetic average finite difference method [17] without fitting factor for small values of $\varepsilon \le h$, which shows that present scheme gives accurate results than three – point arithmetic average finite difference method. It shows the importance of the fitting factor introduced in the scheme. It is observed from the results that the present method approximate the exact solution very well. We have also presented graphical representation of the numerical and exact solution for the problems in Figures 1-4, to show the behaviour of the layer at both ends. From the graphs, we observed that numerical solution approximate exact solution very well in the boundary layers.

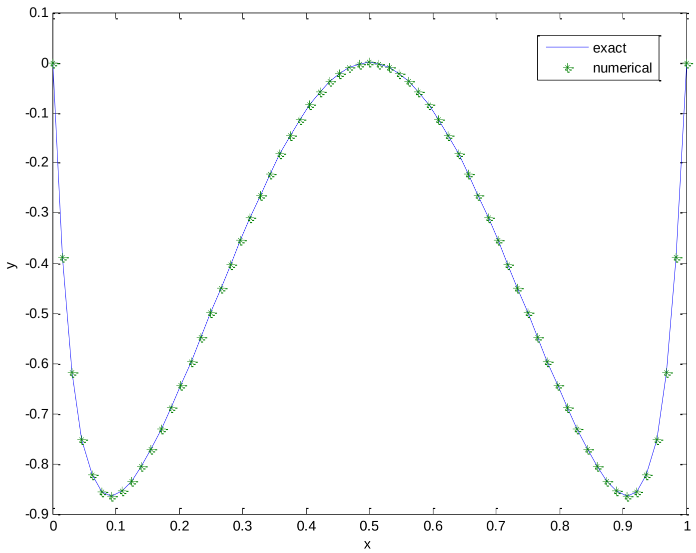

Figure 1. Numerical solution of Example 1 with $h = 2^{-6}$, $\varepsilon = 10^{-3}$

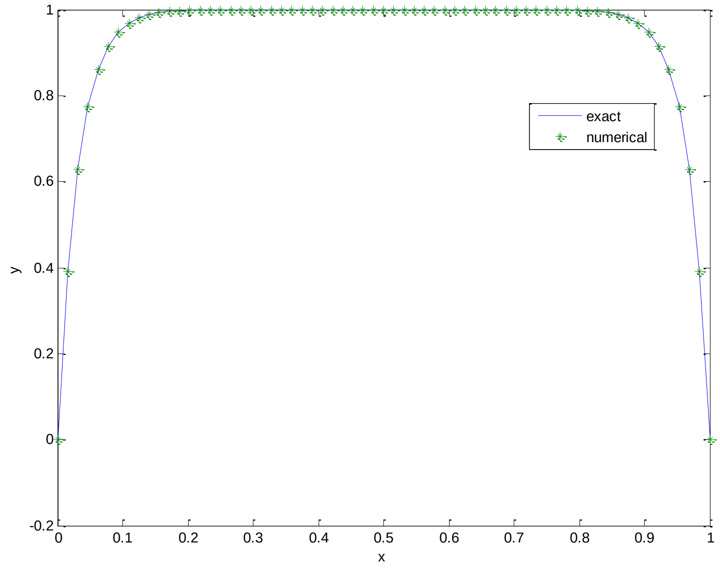

Figure 2. Numerical solution of Example 2 with $h = 2^{-6}, \ \varepsilon = 10^{-3}$

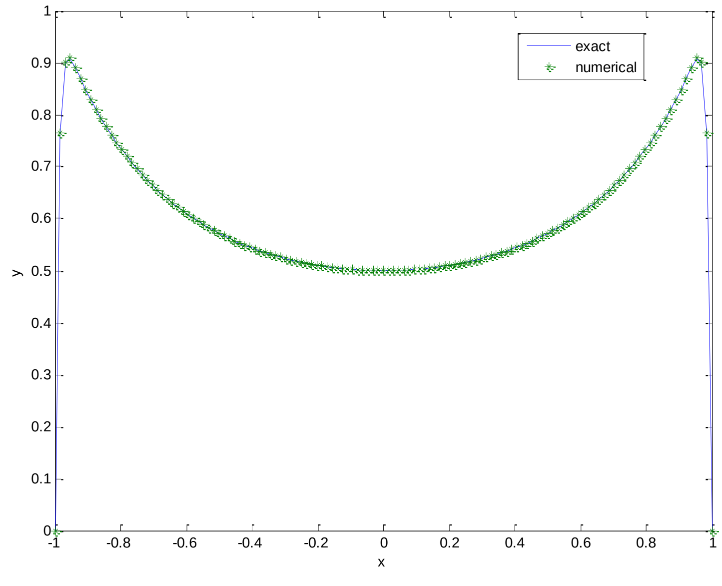

Figure 3. Numerical solution of Example 3 with $h = 2^{-6}, \ \varepsilon = 10^{-4}$

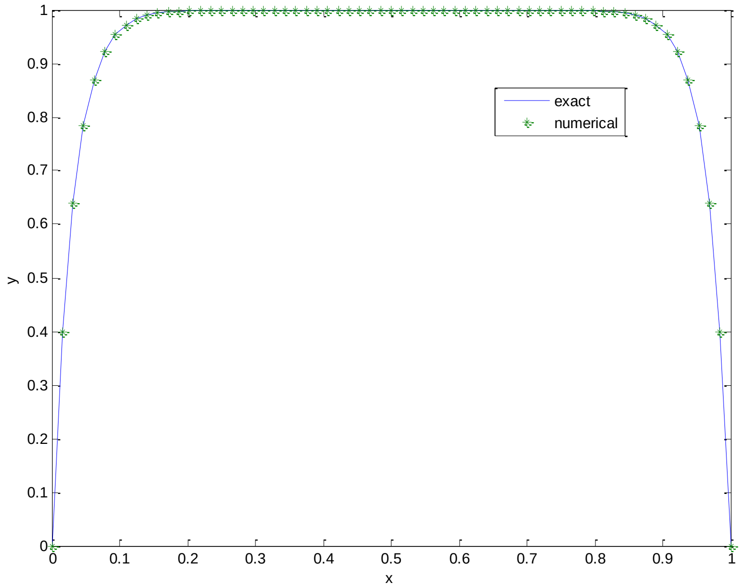

Figure 4. Numerical solution of Example 4 with $h = 2^{-6}$, $\varepsilon = 10^{-3}$

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