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On a Variant of Newton's Method for Simple and Multiple Roots of Non Linear Equations

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Abstract: In this paper the convergence behavior of a variant of Newton's method based on the root mean square for solving nonlinear equations is proposed. The convergence properties of this method for solving non linear equations which have simple or multiple roots have been discussed and it is shown that it converges cubically to simple roots and linearly to multiple roots. Moreover, the values of the corresponding asymptotic error constants of convergence are determined. Theoretical results have been verified on the relevant numerical problems. A comparison of the efficiency of this method with other mean-based Newton's methods is also included. Convergence behavior and error equations are also exhibited graphically for comparison on considering a particular example.

Keywords: Numerical analysis; nonlinear equations; iterative methods; root mean square; order of convergence; asymptotic error constants.

1. Introduction

Solving nonlinear equations is one of the most ubiquitous realistic challenges and interesting task in applied mathematics, numerical analysis, engineering sciences, optimization theory, control theory, economic models and other related disciplines. Finding the exact solution of the nonlinear equation is unlikely, hence the devise of iterative formulae for solving non linear equations of the type.

$$f(x) = 0, \tag{1}$$

which may be higher order algebraic equation and may involve trigonometric, exponential and hyperbolic terms or completely be a transcendental equation is very important and interesting tasks in above stated disciplines. Since finding the exact solution of (1) is very complicated and generally unworkable, therefore to overcome this intricacy numerical methods have been urbanized for finding the numerical solutions such transcendental nonlinear equation. In recent years, various modifications of the Newton method for multiple roots have been proposed and analyzed. However, there are not many methods known to handle the case of multiple roots. Recently many modifications in Newton's method are proposed without using second derivative. A comprehensive class of problems which arise in different other fields [1-4] considered and formulated as a nonlinear equations. Trapezoidal Newton's or arithmetic mean Newton's (AN) method using the trapezoid rule derived by [5]. In [6] modified Newton's method by using Simpson's rule a third order convergent method is obtained. Modifications in the Newton's method to derive the iterative method with efficiency index of 1.442 and the order of convergence of three

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is accomplished in [7]. With the same efficiency index, [8] also derived a new variant of Newton's method. In [9], [10], and [11] nonlinear solvers with closed formulae for multiple roots using iterative methods is studied. Some new iterative formulae having third order convergence is described in [12].

The order of convergence of such methods when we have multiple roots has been by [7] have studied. They have proved that the order of convergence of the modification of the Newton's method go down to one but, when the multiplicity p is known, it may be raised up to two by using two different types of correction. When p is unknown we show that the two most efficient methods in the family of the modification of the Newton's method converge faster than the classical Newton's method. In [13] the convergence behavior of a variant of Newton's method based on the geometric mean is considered. The convergence properties of this method for solving equations which have simple or multiple roots have been discussed and it has been shown that it converges cubically to simple roots and linearly to multiple roots. They have also determined the values of the corresponding asymptotic error constants of convergence. A comparison of the efficiency of the method with other mean-based Newton's methods, based on the arithmetic and harmonic means, is also incorporated. A more detailed survey of these most important techniques, some excellent text book [2] is available in the literature.

In this paper the convergence behavior of a variant of Newton's method based on the root mean square for solving nonlinear equations which have simple or multiple roots is studied.

The algorithm used in classical Newton' s iterative method to estimate the root of a nonlinear equations in one variable is perhaps the most outstanding and widely known, is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2\ldots \tag{2}$$

where x_0 is an initial approximation sufficiently close to α, a root of (1). If α is a simple root of (1), then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and if α is multiple root of equation (1) of multiplicity m, then $f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$. In [14] it is shown that this method converges quadratically to simple roots and the convergence degenerates to linear when approximating a root with multiplicity larger than one and the error equations are given are

$$e_{n+1} = \frac{f''(\alpha)}{2f'(\alpha)} e_n^2 + O(e_n^3), \text{ (for simple roots)} \tag{3}$$

$$e_{n+1} = \left(1 - \frac{1}{m}\right) e_n + O(e_n^2) \text{ (for multiple roots with multiplicity } m\text{)} \tag{4}$$

Some of the **basic definitions** used in such methods are:

Error Equation: If α be a root of the function $f(x) = 0$ and assume that $\{x_{n+1}\}$, and $\{x_n\}$be the consecutive iterations closer to the root α and$\{e_{n+1}\}$, and $\{e_n\}$are errors in these iterations, then $e_n = x_n - \alpha$ be the error in the nth iteration. We call the relation

$$e_{n+1} = Ce_n{}^p + O(\, e_n{}^{p+1}) \tag{5}$$

as the error equation. If we obtain the error equation for any iterative method, then the value of p is its order of convergence.

Convergence and Order of Convergence: Let $f(x)$ be a real function with a simple root α and $\{x_n\}, n \geq 0$ be a sequence of real numbers, converging towards α. Then, we say that the order of convergence of the sequence is p, if there exists a real $p \geq 1$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C \tag{6}$$

for some C $\neq$ 0, C is known as the asymptotic error constant (AEC). If p=1, 2, or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Computational Order of Convergence (COC): Let α be a root of the function $f(x)$ and $\{x_{n+1}\}, \{x_n\}$ and $\{x_{n-1}\}$ be three consecutive iterations closer to the root α. Then the computational order of convergence (COC) can be approximated by the formula

$$COC \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{7}$$

Efficiency Index: Let r be the number of new pieces of information required by a method. A "piece of information" ıs typically any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index and is defined by Efficiency Index $\rho = \sqrt[r]{p}$ where p is the order of the method and *r* is the number of functions evaluation required by the method.

2. Description of the Root Mean Square Newton's Method

Let α be a simple zero of a sufficiently differentiable function $f: I \subset R \to R$ $f(x)=0$. From Newton's theorem, we have

$$f(x) = f(x_n) + \int_{x_n}^{x} f(t)dt. \tag{8}$$

Approximate the above definite integral by $(x-x_n)f(x_n)$ and take $x=\alpha$, we have

$$0 \approx f(x_n) + (\alpha - x_n) f'(x_n) \tag{9}$$

Now approximate x_{n+1} toα is given by the best known numerical method for solving $f(x)=0$ as the classical Newton's method (CN) and is expressed by (1). An improvement to the iteration of Newton's method and approximated the indefinite integral by a trapezoid instead of a rectangle and the results a new method with third-order convergence, they proposed the following variant have suggested in [15]:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_{n+1})}, \tag{10a}$$

$$z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,\cdots, \tag{10b}$$

It is termed as arithmetic mean Newton's method (AN).This method has third-order convergence. Harmonic Mean Newton's method (HN): In equation (5), if we use the harmonic mean instead of arithmetic mean, we obtain harmonic mean Newton's method [6].The harmonic means instead of the arithmetic mean considered in (5), and obtain harmonic mean Newton's method (HN) by [8]

$$x_{n+1}=x_n-\frac{f(x_n)\left[f'(x_n)+f'(z_{n+1})\right]}{2f'(x_n)f'(z_{n+1})}. \tag{11}$$

In the midpoint Newton's (MN) method the integral in (3) is approximated using the midpoint integration rule instead of trapezoidal rule, and obtain the midpoint Newton's method [8]

$$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\dfrac{x_n+z_{n+1}}{2}\right)}, \tag{12}$$

Similarly, Ref. [13] used the geometric mean instead of the arithmetic mean and proposed and discussed Geometric mean Newton's method (GN) mean instead of the arithmetic mean and proposed and discussed Geometric mean Newton's method

$$x_{n+1}=x_n-\frac{f(x_n)}{signf(x_0)\sqrt{f'(x_n)f'(z_{n+1})}}. \tag{13a}$$

Logarithmic mean instead of the arithmetic mean is used by [16] and obtain the following scheme:

$$x_{n+1}=x_n-f(x_n)\frac{\left(\ln\left|f'(z_{n+1})\right|-\ln\left|f'(x_n)\right|\right)}{f'(z_{n+1})-f'(x_n)},\ z_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\quad n=0,1,2\cdots \tag{13b}$$

In this article, modifications of the Newton method for simple and multiple roots using Root Mean Square Newton's method (RMN) have been proposed and analyzed using the following scheme:

$$x_{n+1}=x_n-\frac{f(x_n)}{sign\,f'(x_0)}\sqrt{\frac{2}{\left[f'^2(x_n)+f'^2(z_{n+1})\right]}},\ n=0,1,2\cdots \tag{14}$$

$$z_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},n=0,1\cdots$$

We call this scheme as Root mean square Newton's method (RMN) proposed in [2].Since the techniques mentioned above uses the mean computation, they are members of a family of Mean-Based Newton's methods (MBN).

3. Convergence Analysis

The behavior of the convergence of this Root Mean Square Newton's method (RMN) method for simple and multiple roots is considered in the following theorems.

Theorem 1: If $\alpha\in I$ be a simple zero of a sufficiently differentiable function $f:I\to R$ for an open interval I. If $f(x)$ is sufficiently smooth in the neighborhood of α and the initial value x_0 then the CN methods defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{sign\,f'(x_0)}\sqrt{\frac{2}{\left[f'^2(x_n)+f'^2(z_{n+1})\right]}},\quad n=0,1,2\cdots$$

converges cubically and satisfies the following error equation:

$$e_{n+1}=\left(3c_2^2+c_3\right)\frac{1}{2}e_n^3+O\left(e_n^4\right)$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3$

Proof: Let α be a simple zero of $f(x)$. Since $f(x)$ sufficiently differentiable function, expanding $f(x_n)$ and $f'(x_n)$ about α, we get

$$f(x_n) = f(\alpha) + f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f'''(\alpha)e_n^3 + \cdots$$

$$= f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + \cdots\right] \text{(Recall that } f(\alpha) = 0\text{) and} \tag{15}$$

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots\right] \tag{16}$$

Where

$$e_n = x_n - \alpha \quad \text{and} \quad c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3\cdots \tag{17}$$

Using (15) and (16) and Simplifying, we have

$$z_{n+1} = \alpha + c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) \tag{18}$$

$$\because \frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O\left(e_n^4\right)$$

Again using (16) and (18), we obtain

$$f'(z_{n+1}) = f'(\alpha) + \left(z_{n+1} - \alpha\right)f''(\alpha) + \frac{1}{2!}\left(z_{n+1} - \alpha\right)^2 f'''(\alpha) + \cdots$$

$$= f'(\alpha) + \left(c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4)\right)f''(\alpha) + O(e_n^4) + \cdots$$

$$= f'(\alpha)\left[1 + c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4)\right] \tag{19}$$

$$f'(x_n)^2 = f'(\alpha)^2\left[1 + 4c_2e_n + 2\left(3c_3 + 2c_2^2\right)e_n^2 + 4\left(2c_4 + 3c_2c_3\right)e_n^3 + \cdots\right] \tag{20}$$

$$f'(z_{n+1})^2 = f'(\alpha)^2\left[1 + 4c_2^2e_n^2 + 8\left(c_2c_3 - c_2^2\right)e_n^3 + \cdots\right] \tag{21}$$

Adding (20) and (21) we have

$$f'^2(x_n) + f'^2(z_{n+1}) = 2f'^2(\alpha)\left[1 + 2c_2e_n + \left(4c_2^2 + 3c_3\right)e_n^2 + \left(10c_2c_3 - 4c_2^3 + 4c_4\right)e_n^3 + O(e_n^4)\right] \tag{22}$$

Further, we have

$$\left[f'^2(x_n) + f'^2(z_{n+1})\right]^{\frac{1}{2}} = \left\{2f'^2(\alpha)\left[1 + 2c_2e_n + \left(3c_3 + 4c_2^2\right)e_n^2 + (10c_2c_3 - 4c_2^3 + 4c_4)e_n^3 + O(e_n^4)\right]\right\}^{\frac{1}{2}} \tag{23}$$

Now using (23) and simplify

$$\frac{f(x_n)2^{\frac{1}{2}}}{sign\, f'(x_0)sqrt\left[f'^2(x_n) + f'^2(z_{n+1})\right]} = f'(x_n)[2]^{\frac{1}{2}}\left[f'^2(x_n) + f'^2(z_{n+1})\right]^{-\frac{1}{2}} = e_n + \left(-\frac{3}{2}c_2^2 - \frac{1}{2}c_3\right)e_n^3 + O(e_n^4)$$

$$= e_n - \left(\frac{1}{2}c_3 + \frac{16}{3}c_2^2\right)e_n^3 + O(e_n^4) \tag{24}$$

Hence

$$x_{n+1} = x_n - f(x_n)\frac{2^{\frac{1}{2}}}{\left[f'^2(x_n) + f'^2(z_{n+1})\right]^{\frac{1}{2}}},$$

$$\Rightarrow x_{n+1} = x_n - \left[e_n + \left(-\frac{3}{2}c_2^2 - \frac{1}{2}c_3\right)e_n^3 + O(e_n^4)\right]$$

on subtracting α from both sides of this equation we get ***error equation***

$$e_{n+1} = e_n - \left[e_n + \left(-\frac{3}{2}c_2^2 - \frac{1}{2}c_3\right)e_n^3 + O(e_n^4)\right] e_{n+1} = \left(\frac{3}{2}c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4) \tag{25}$$

The asymptotic range (for $n \to \infty$), the higher-order terms in above equation are neglected. The "asymptotic equality" is denoted by $\approx$ and is used when the equality holds for $n \to \infty$. Then

$$e_{n+1} \approx \left(\frac{3}{2}c_2^2 + \frac{1}{2}c_3\right)e_n^3, \tag{26}$$

$$\Rightarrow \lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^3} = \left(\frac{3}{2}c_2^2 + \frac{1}{2}c_3\right) , \tag{27}$$

which shows that root mean square Newton's method is of third order and hence converges cubically for simple root .

Theorem 2: If $\alpha \in I$ be a multiple zero of multiplicity m (>1) of a sufficiently differentiable function $f : I \to R$ for an open interval I . If $f(x)$ is sufficiently smooth in the neighborhood of α , then the root mean square Newton (RMN) method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{sign\, f'(x_0)}\sqrt{\frac{2}{\left[f'^2(x_n) + f'^2(z_{n+1})\right]}},\ n = 0,1,2\cdots$$

converges linearly to α and satisfies the following error equation:

$$e_{n+1} = \left(1 - \frac{\sqrt{2}}{m\sqrt{1+G^2}}\right)e_n - \frac{\sqrt{2}(1+G^2)}{2m}\left[c_2 - \frac{\left(c_2(m+1) + G^2\left(\frac{m^2-1}{2m^2}\right)\Big/ m^2\right)}{(1+G^2)}\right]e_n^2 + O\left(e_n^3\right)$$

where $e_n = x_n - \alpha$ and $c_2 = \dfrac{1}{2!}\dfrac{f^{(2)}(\alpha)}{f'(\alpha)}$, $G = \left(1 - \dfrac{1}{m}\right)e_n + c_2\dfrac{1}{m^2}e_n^2 + O\left(e_n^3\right)$

Proof: Let α be root of equation $f(x) = 0$ **,** of multiplicity m .

Then $f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$

By Taylor expansion of $f(x_n)$ about α , we have

$$f(x_n) = \frac{f^{(m)}(a)}{m!}e_n^m + \frac{f^{(m+1)}(a)}{(m+1)!}e_n^{m+1} + \frac{f^{(m+2)}(a)}{(m+2)!}e_n^{m+2} + O\left(e_n^{m+3}\right) \tag{28}$$

Rewriting the above equation as

$$= \frac{f^{(m)}(a)}{m!} e_n^m \left[1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3 + O\left(e_n^4\right)\right] \tag{29}$$

where

$$e_n = x_n - \alpha,\ c_j = \frac{f^{(m+j-1)}(a)}{f^{(m)}(a)(m+1)(m+2)\cdots(m+j-1).}\ ,\quad j = 2,\ 3\cdots \tag{30}$$

Similarly from (29), we have

$$f'(x_n) = \frac{f^{(m)}(a)}{(m-1)!} e_n^{m-1} \left[1 + c_2 \frac{m+1}{m} e_n + c_3 \frac{m+2}{m} e_n^2 + O\left(e_n^3\right)\right] \tag{31}$$

Now on using (29) and (31) and simplifying, we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{\frac{f^{(m)}(a)}{m!} e_n^m \left[1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3 + O\left(e_n^4\right)\right]}{\frac{f^{(m)}(a)}{(m-1)!} e_n^{m-1} \left[1 + c_2 \frac{m+1}{m} e_n + c_3 \frac{m+2}{m} e_n^2 + O\left(e_n^3\right)\right]}$$

$$= \frac{1}{m}\left[e_n + c_2 e_n^2 + c_3 e_n^3 + O\left(e_n^4\right)\right]\left[1 + c_2 \frac{m+1}{m} e_n + c_3 \frac{m+2}{m} e_n^2 + O\left(e_n^3\right)\right]^{-1} \tag{32}$$

Now using (32) and $x_n = e_n + \alpha$ in above relation, we have

$$z_{n+1} = \alpha + \left(1 - \frac{1}{m}\right) e_n + c_2 \frac{1}{m^2} e_n^2 + O\left(e_n^3\right), \tag{33}$$

where

$$z_{n+1} = \alpha + G,\ G = \left(1 - \frac{1}{m}\right) e_n + c_2 \frac{1}{m^2} e_n^2 + O\left(e_n^3\right) \tag{34}$$

On expanding $f'(z_{n+1})$ about α and using (31) we obtain

$$\Rightarrow f'(z_{n+1}) = \frac{f^{(m)}(a)}{(m-1)!} G_n^{m-1} \left[1 + c_2 \frac{m+1}{m} G + c_3 \frac{m+2}{m} G^2 + O\left(G^3\right)\right]$$

On using (34) in above equation and simplifies to

$$\therefore f'(z_{n+1}) = G \frac{f^{(m)}(a)}{(m-1)!} e_n^{m-1} \left[1 + c_2 \frac{m+1}{m} \left(\left(1 - \frac{1}{m}\right) e_n + c_2 \frac{1}{m^2} e_n^2\right) + c_3 \frac{m+2}{m} \left(\left(1 - \frac{1}{m}\right) e_n + c_2 \frac{1}{m^2} e_n^2\right)^2 + O\left(e_n^3\right)\right] \tag{35}$$

Now

$$f'(x_n)^2 = \left(\frac{f^{(m)}(a)}{(m-1)!} e_n^{m-1}\right)^2 \left[1 + \left(2c_2 \frac{m+1}{m}\right) e_n + \left(\left(c_2 \frac{m+1}{m}\right)^2 + \left(2c_3 \frac{m+2}{m}\right)\right) e_n^2 + O\left(e_n^3\right)\right] \tag{36}$$

$$f'(z_{n+1})^2 = \left(G\frac{f^{(m)}(a)e_n^{m-1}}{(m-1)!}\right)^2 \begin{bmatrix} 1+\dfrac{\left(m^2-1\right)}{m^2}e_n \\ +\left[2(m+2)\dfrac{\left(m^2-1\right)}{m^3}c_3+c_2^2\dfrac{(m+1)(m^3-m^2+m+1)}{m^4}\right]e_n^2+O\left(e_n^3\right) \end{bmatrix} \quad (37)$$

Adding (36) and (37) we have

$$f'^2(x_n)+f'^2(z_{n+1})$$

$$=\left(\frac{f^{(m)}(a)e_n^{m-1}}{(m-1)!}\right)^2 \begin{bmatrix} \left(1+G^2\right)+\left(2c_2\dfrac{m+1}{m}+G\dfrac{\left(m^2-1\right)}{m^2}\right)e_n \\ +\left(\left(c_2\dfrac{m+1}{m}\right)^2+2c_3\dfrac{m+2}{m}+G^2\begin{pmatrix} 2(m+2)\dfrac{(m-1)^2}{m^3}c_3 \\ c_2^2\dfrac{(m+1)(m^3-m^2+m+1)}{m^4}\end{pmatrix}\right)e_n^2+O\left(e_n^3\right) \end{bmatrix} \quad (38)$$

Now using (38), we have

$$\frac{f(x_n)}{sign\, f'(x_0)}\sqrt{\frac{2}{\left[f'^2(x_n)+f'^2(z_{n+1})\right]}},$$

$$=\sqrt{2}\left(\frac{1}{m\sqrt{1+G^2}}e_n\right)\begin{bmatrix} 1+\left(c_2-\dfrac{c_2(m+1)+\frac{1}{2}G^2\left(\frac{m^2-1}{m^2}\right)}{m}\right)\dfrac{e_n}{1+G^2} \\ +\begin{bmatrix} c_3-\dfrac{c_2(m+1)+\frac{1}{2}G^2\left(\frac{m^2-1}{m^2}\right)}{1+G^2}c_2 \\ -\dfrac{\left\{\dfrac{(m+1)^2}{m^2}c_2^2+\dfrac{2}{m}c_3(m+2)+G^2\begin{pmatrix} 2(m+2)\dfrac{(m-1)^2}{m^3}c_3 \\ +c_2^2\dfrac{(m+1)(m^3-m^2+m+1)}{m^4}\end{pmatrix}\right\}}{2\left(1+G^2\right)} \\ +\dfrac{3\left[2c_2\dfrac{(m+1)}{m}+G^2\left(\dfrac{m^2-1}{m^2}\right)\right]^2}{8\left(1+G^2\right)} \end{bmatrix}e_n^2+O\left(e_n^3\right) \end{bmatrix} \quad (39)$$

Now using (39) with in the following equation, we have

$$x_{n+1}=x_n-\frac{f(x_n)}{sign\, f'(x_0)}\sqrt{\frac{2}{\left[f'^2(x_n)+f'^2(z_{n+1})\right]}},$$

$$x_{n+1}=x_n-=\sqrt{2}\left(\frac{1}{m\sqrt{1+G^2}}e_n\right)\left[\begin{array}{l}1+\left(c_2-\dfrac{c_2(m+1)+\frac{1}{2}G^2\left(\frac{m^2-1}{m^2}\right)}{m}\right)\dfrac{e_n}{1+G^2}\\ +\left[\begin{array}{l}c_3-\dfrac{c_2(m+1)+\frac{1}{2}G^2\left(\frac{m^2-1}{m^2}\right)}{1+G^2}c_2\\ -\dfrac{\left\{\frac{(m+1)^2}{m^2}c_2^2+\frac{2}{m}c_3(m+2)+G^2\left(\begin{array}{l}2(m+2)\frac{(m-1)^2}{m^3}c_3\\ +c_2^2\frac{(m+1)(m^3-m^2+m+1)}{m^4}\end{array}\right)\right\}}{2\left(1+G^2\right)}\\ +\dfrac{3\left[2c_2\frac{(m+1)}{m}+G^2\left(\frac{m^2-1}{m^2}\right)\right]^2}{8\left(1+G^2\right)}\end{array}\right]e_n^2+O\left(e_n^3\right)\end{array}\right]$$

On Subtracting α from both sides of (39), and on simplification, we have

$$e_{n+1}=\left(1-\frac{\sqrt{2}}{m\sqrt{1+\mathrm{G}^2}}\right)e_n-\frac{\sqrt{2}(1+\mathrm{G}^2)}{2m}\left[c_2-\frac{\left(c_2(m+1)+G^2\left(\frac{m^2-1}{2m^2}\right)\Big/m^2\right)}{(1+\mathrm{G}^2)}\right]e_n^2+O\left(e_n^3\right) \tag{40}$$

$$e_{n+1}=\left(1-\frac{\sqrt{2}}{m\sqrt{1+\mathrm{G}^2}}\right)e_n-\frac{\sqrt{2}(1+\mathrm{G}^2)}{2m}\left[c_2-\frac{\left(c_2(m+1)+G^2\left(\frac{m^2-1}{2m^2}\right)\Big/m^2\right)}{(1+\mathrm{G}^2)}\right]e_n^2+O\left(e_n^3\right)$$

Then, in the asymptotic range,

$$e_{n+1}\approx\left(1-\frac{\sqrt{2}}{m\sqrt{1+\mathrm{G}^2}}\right)e_n \tag{41}$$

Consequently, for roots of multiplicity *m* the method exhibits linear convergence with asymptotic error constant (AEC) equal to $\left(1-\dfrac{\sqrt{2}}{m\sqrt{1+\mathrm{G}^2}}\right)$.

This method is third-order convergent for simple roots and its efficiency index is $\sqrt[3]{3} = 1.442$. Consequently, for roots of multiplicity m, method exhibits the linear convergence with which (41) is the error equation of the using root mean square (RMN) method. The convergence properties of Newton's and the considered MBN methods are summarized in Table 1. For simple roots the MBN methods have cubic order of convergence, but the Classical Newton (CN) method has 'only' quadratic order of convergence. The analysis also proves that the MBN methods lose the cubically order of convergence when the root has multiplicity m > 1. In that case the order of the MBN methods is linear as the same as the order of convergence of the CN method. However, the AEC for the MBN methods is less than the AEC for the CHN method, which ensures better convergence in case of multiple roots, too.

4. Numerical Results and Discussion

In this section, we present the results of some numerical tests to compare the efficiencies of the new root mean square Newton's method (RMN) with Newton method (CN). We employed (CN), (AN) [7], (HN) and (MN) methods in [2] and GN method of [13]. Numerical computations reported here have been carried out in a MathCad environment. The stopping criterion has been taken as $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-7}$, and the following test functions have been used.

Table 1. Convergence properties of Newton's and the considered MBN methods

Multiplicity of root (m)	Order of Convergence					Asymptotic Error Constants (AEC)				
	CN	AN	HN	GN	**RMN**	CN	AN	HN	GN	**RMN**
$m=1$	**2**	**3**	**3**	**3**	**3**	c_2	$\frac{1}{2}\left(c_2^2+c_3\right)$	$\frac{1}{2}c_3$	$\frac{c_2^2}{4}+\frac{c_3}{2}$	$\frac{3}{2}c_2^2+\frac{1}{2}c_3$
$m=2$	1	1	1	1	**1**	0.5	0.333	0.25	0.293	**0.368**
$m=3$	1	1	1	1	**1**	0.667	0.538	0.458	0.5	**0.569**
$m=4$	1	1	1	1	**1**	0.75	0.648	0.579	0.615	**0.674**
$m=5$	1	1	1	1	**1**	0.8	0.716	0.656	0.688	**0.738**
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$m=\lambda$	**1**	**1**	**1**	**1**	**1**	$1-\frac{1}{\lambda}$	$1-\frac{2}{(1+G)\lambda}$	$1-\frac{1+\frac{1}{G}}{2\lambda}$	$1-\frac{1}{\lambda\sqrt{G}}$	$1-\frac{\sqrt{2}}{\lambda\sqrt{1+G^2}}$

NM - Newton's method; AM - Arithmetic mean Newton method; HM - Harmonic mean Newton method; GM - Geometric mean Newton method; RMN- root mean square; COC - Computational order of convergence; NOFE - Number of functional evaluations; *i* - Number of iterations; ND - Not defined

By the numerical results in from the Table 2 it is evident that the total number of functional evaluations required for the CHN method is mostly less than in the CN and AN methods, but greater than in the HN method. These differences are higher when we approximate multiple roots. However, for the function in (a) for x 0 = 0.1 the NOFE required for the CHN method is the least, which shows the importance of the initial approximations and the properties of the functions.

Table 2. Computational order of convergence, numbers of iterations and other properties of Newton's and the considered MBN methods

$f(x)=0$	x_0	m	COC					i					NOEF				
			CN	AN	HN	GN	**RMN**	CN	AN	HN	GN	**RMN**	CN	AN	HN	GN	**RMN**
$f_1(x)$	0.1	1	2.00	ND	ND	3.02	**3.04**	9	8	5	4	**6**	18	24	15	12	**12**
	2	1	2.00	ND	ND	ND	**ND**	4	3	3	3	**4**	8	12	12	12	**8**
$f_2(x)$	-2	1	2.00	3.00	3.01	3.00	**2.994**	8	5	4	5	**6**	16	15	12	15	**18**
	-3	1	2.00	ND	3.01	3.00	**2.990**	13	9	7	8	**10**	26	27	21	24	**30**
$f_3(x)$	-1	1	2.00	ND	ND	3.01	**3.093**	5	3	3	3	**4**	10	12	9	9	**12**
	-3	1	2.00	2.03	ND	ND	**2.980**	5	3	3	3	**4**	10	9	9	9	**12**
$f_4(x)$	-1.5	1	1.99	3.01	3.09	2.84	**2.990**	4	3	3	3	**4**	8	9	9	9	**12**
	1	2	1.00	1.00	1.00	1.00	**ND**	23	14	11	13	**24**	46	42	33	39	**72**
	3	2	1.00	1.00	1.00	1.00	**1.00**	24	15	12	14	**17**	48	45	36	42	**51**
$f_5(x)$	-0.8	1	2.00	3.28	3.20	3.29	**ND**	4	3	2	2	**4**	8	9	6	6	**12**
	0.2	2	1.00	1.00	1.00	1.00	**1.00**	23	14	12	13	**7**	46	42	36	39	**21**
	1.2	2	1.00	1.00	1.00	1.00	**1.00**	22	14	11	12	**16**	42	42	33	36	**48**
$f_6(x)$	-0.3	3	1.00	1.00	1.00	1.00	**1.00**	37	24	19	22	**26**	74	72	57	66	**78**
	0.4	3	1.00	1.00	1.00	1.00	**1.00**	37	24	19	22	**26**	74	72	57	66	**78**
$f_7(x)$	1.4	3	1.00	1.00	1.00	1.00	**1.00**	38	25	20	22	**23**	76	75	60	66	**69**
	-3	4	1.00	1.00	1.00	1.00	**1.00**	57	38	30	34	**34**	114	114	90	102	**102**
$f_8(x)$	2.6	4	1.00	1.00	1.00	1.00	**1.00**	57	38	30	34	**37**	114	114	90	102	**111**
	3.01	4	1.00	1.00	1.00	1.00	**3.051**	41	27	22	24	**24**	82	81	66	72	**72**

It is interesting to see that for equation $f_8(x)=0$ when the initial approximation taken as $x_0=2.6$ or $x_0=3.01$ in Geometric Mean Newton Method (GN) then $\{x_n\}\to 3$, whereas in root mean square Newton's method (RMN) $\{x_n\}\to 3$ when $x_0=2.6(=\alpha)$ and $\{x_n\}\to 6\ (=\alpha)$ when $x_0=3.01$. This is also illustrated in Figure 1 and Figure 2.**Test** functions ([13]) and the stopping criterion are $|x_{n+1}-\alpha|+|f(x_{n+1})|<10^{-7}$

(a) $f_1(x)=x^3+4x^2-10=0$, $\alpha=1.365230013414097$,

(b) $f_2(x)=xe^{x^2}-\sin^2 x+3\cos x+5=0$, $\alpha=-1.207647827130919$,

(c) $f_3(x)=sin^2x-x^2+1=0$, $\alpha=-1.404491648215341$,

(d) $f_4(x)=(x-2)^2(x+1)=0$, $\alpha=-1$ or $\alpha=2$

(e) $f_5(x)=\left(\sin x-\frac{\sqrt{2}}{2}\right)^2(x+1)=0$, $\alpha=\frac{\pi}{4}$ or $\alpha=-1$

(f) $f_6(x)=x^2\sin(4x)=0$, $\alpha=0$

(g) $f_7(x)=(x-2)^3(x+2)^4=0,\quad \alpha=2$ or $\alpha=-2$

(h) $f_8(x)=\ln^2(x-2)(e^{x-3}-1)\sin\left(\frac{\pi x}{3}\right)=0,\ \alpha=3$

Convergence and error propagation, in Root Mean Square Newton method (RMN).

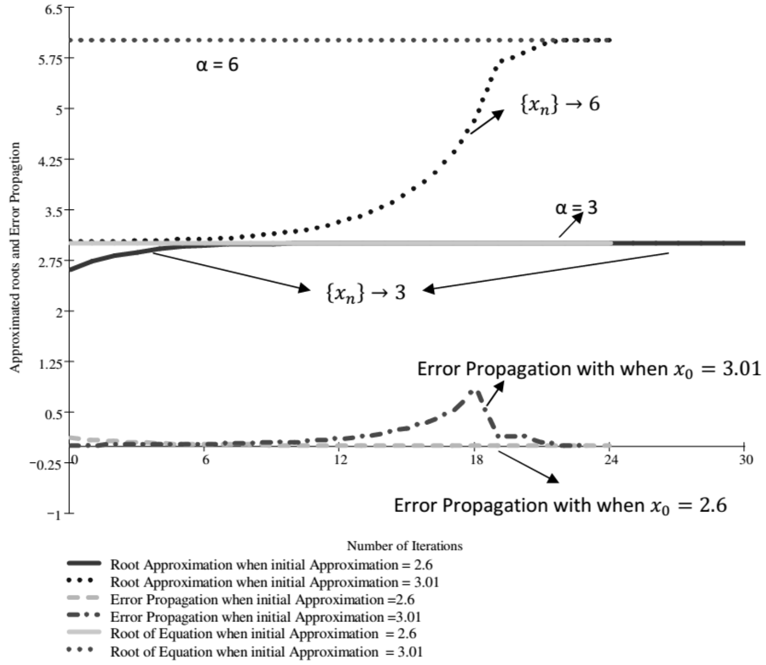

Figure 1. Convergence and error propagation of $\ln^2(x-2)(e^{x-3}-1)\sin\left(\frac{\pi x}{3}\right)=0$, with initial approximations, $x_0 = 2.6$ and $x_0 = 3.01$, using Root Mean Square Newton method (RMN).

Convergence and error propagation, in Geometric Mean Newton method (GN).

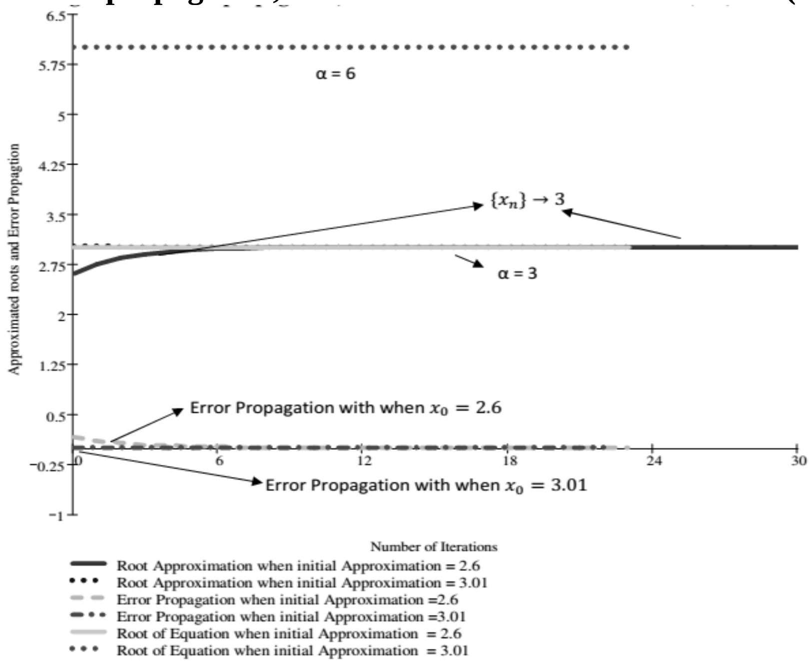

Figure 2. Convergence and error propagation of $\ln^2(x-2)(e^{x-3}-1)\sin\left(\frac{\pi x}{3}\right)=0$, with initial approximations, $x_0 = 2.6$ and $x_0 = 3.01$, using Geometric Mean Newton method (GN).

All numerical tests agree with the theoretically results of paper. The most important characteristics of Root Mean Square method (RMN) are: (1) third order of convergence (for simple roots). (2) Does not require the computation of second or higher order derivatives. (3) By the numerical results (Table 1.) it is evident that the total number of functional evaluation required is less than of Newton's method. It is interesting to consider the behavior of tested methods for multiple roots. The test function (f) has a multiple roots and the COC is linear. This is in accordance with the theoretically properties of Newton's method for multiple roots**.** Table 1: Convergence properties of Newton’s and the considered MBN methods. Table 2. Computational order of convergence, numbers of iterations and other properties of Newton’s and the considered MBN methods. Convergence and error propagation in Root Mean Square Newton method (RMN) are shown graphically.

Results obtained in this paper are useful to understand the convergence properties while solving non linear equations obtained at the same time of modeling the research problems mathematically, which have simple or multiple roots. The method discussed the computational techniques of solving equations with simple or multiple roots which does not does not require the computation of second or higher order derivatives, while solving numerically.

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