# Fourth Order Numerical Method for Singularly Perturbed Delay Differential Equations 

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#### Abstract

This paper presents a numerical method to solve singularly perturbed delay differential equations. The solution of this problem exhibits layer or oscillatory behaviour depending on the sign of the sum of coefficients in reaction terms. A fourth order finite difference scheme on a uniform mesh is developed. The stability and convergence of the proposed method have been established. The effect of delay parameter (small shift) on the boundary layer(s) has also been analyzed and depicted in graphs. The applicability of the proposed scheme is validated by implementing it on four model examples. To show the accuracy of the method, the results are presented in terms of maximum absolute errors.


Keywords: Singular perturbation; delay differential equation; boundary layer.

## 1. Introduction

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay team. The smoothness of the solutions of such singularly perturbed delay differential equation deteriorates when the parameter tends to zero. Such problems arise frequently in the study of control theory [1], in determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in dendrites [2], in the modeling of activation of a neuron [3] and many more. A well-known fact is that the solution of such problems display sharp boundary or interior layers when $\varepsilon$ is very small, i.e., the solution varies rapidly in some parts and varies slowly in some other parts. So the treatment of singularly perturbed problems (SPPs) presents severe difficulties that have to be addressed to ensure accurate numerical solutions [4-6]. Thus more efficient but simpler computational techniques are required to solve SPPs.

Recently, some researchers are tried to develop a numerical methods for solving singularly perturbed delay differential equations. For examples, Gemechis et al. [7] presented numerical solution of singularly perturbed delay reaction-diffusion equations with layer or oscillatory behavior, earlier researchers [8-10] develop some numerical methods to solve singularly perturbed delay differential equations. Kadalbajoo and Ramesh [11] states that, the accuracy of the problem increased by increasing the resolution of the grid which might be impractical in some cases like higher dimensions. Pratima and Sharma [12] states, till date $\varepsilon$-uniformly convergent methods have not been sufficiently developed for a wide class of singularly perturbed delay differential equations. However, this paper present a uniformly convergent and more accurate fourth order numerical method for solving singularly perturbed delay reaction-diffusion equations.

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## 2. Description of the method

Consider a linear singularly perturbed delay reaction-diffusion equation of the form:

$$
\begin{equation*}
\varepsilon y^{\prime \prime}(x)+a(x) y(x-\delta)+b(x) y(x)=f(x), \quad x \in[0,1] \tag{1}
\end{equation*}
$$

with the interval and boundary conditions,

$$
\begin{equation*}
y(x)=\phi(x),-\delta \leq x \leq 0 \text { and } y(1)=\beta \tag{2}
\end{equation*}
$$

where $\varepsilon$ is a perturbation, $0<\varepsilon \ll 1$ and $\delta$ is also a delay parameter, $0<\delta \ll 1$; $a(x), b(x), f(x)$ and $\phi(x)$ are bounded smooth functions in $(0,1)$ and $\beta$ is a given constant. The layer or oscillatory behaviour of the problem under consideration is maintained for $\delta \neq 0$ but sufficiently small, depending on the sign of $a(x)+b(x)$, for all $x \in(0,1)$. If $a(x)+b(x)<0$, the solution of the problem in Equations (1) and (2) exhibits layer behaviour, and if $a(x)+b(x)>0$, it exhibits oscillatory behaviour. Therefore, if the solution exhibits layer behaviour, there will be two boundary layers which will occur at both end points $x=0$ and $x=1$ (see Reference [7]).
By using Taylor series expansion in the neighborhood of the point $x$, we have:

$$
\begin{equation*}
y(x-\delta) \approx y(x)-\delta y^{\prime}(x)+\mathrm{o}\left(\delta^{2}\right) \tag{3}
\end{equation*}
$$

Substituting Equation (3) into Equation (1), we obtain an asymptotically equivalent singularly perturbed two point boundary value problem of the form:

$$
\begin{equation*}
L y(x) \equiv y^{\prime \prime}(x)+p(x) y^{\prime}(x)+q(x) y(x)=r(x) \tag{4}
\end{equation*}
$$

under the boundary conditions,

$$
\begin{equation*}
y(0)=\phi_{0} \text { and } y(1)=\beta \tag{5}
\end{equation*}
$$

where, $p(x)=\frac{-\delta a(x)}{\varepsilon}, \quad q(x)=\frac{a(x)+b(x)}{\varepsilon}$ and $r(x)=\frac{f(x)}{\varepsilon}$.
The transition from Equation (1) to Equation (4) is admitted, because of the condition that $0<\delta \ll 1$ is sufficiently small. Further details on the validity of this transition can be found in Reference [13].

Now, dividing the interval [0,1] into $N$ equal parts with constant mesh length $h$, we have $x_{i}=x_{0}+i h, i=0,1,2, \mathrm{~K}, N$. Let $y_{i}=y\left(x_{i}\right)$ for $x_{i} \in[0,1]$.
Assuming that $y(x)$ has continuous derivatives on [0,1] and making use of Taylor's series expansions of $y_{i+1}$ and $y_{i-1}$ up to $O\left(h^{7}\right)$, we get the finite difference approximations for $y_{i}^{\prime}$ and $y_{i}^{\prime \prime}$ as:

$$
\begin{equation*}
y_{i}^{\prime}=\frac{y_{i+1}-y_{i-1}}{2 h}-\frac{h^{2}}{6} y_{i}^{\prime \prime \prime}-\frac{h^{4}}{120} y_{i}^{(5)}+\tau_{1} \tag{6}
\end{equation*}
$$

where, $\tau_{1}=-\frac{h^{6}}{7!} y^{(7)}\left(\xi_{1}\right)$, for $\xi_{1} \in\left[x_{i-1}, x_{i}\right]$.
and

$$
\begin{equation*}
y_{i}^{\prime \prime}=\frac{y_{i+1}-2 y_{i}+y_{i-1}}{h^{2}}-\frac{h^{2}}{12} y_{i}^{(4)}-\frac{h^{4}}{360} y_{i}^{(6)}+\tau_{2} \tag{7}
\end{equation*}
$$

where, $\tau_{2}=-\frac{h^{6}}{8!} y^{(8)}\left(\xi_{2}\right)$, for $\xi_{2} \in\left[x_{i-1}, x_{i}\right]$.
Substituting Equations (6) and (7) into Equation (4) and simplifying, we obtain:

$$
\begin{equation*}
\frac{1}{h^{2}}\left(y_{i+1}-2 y_{i}+y_{i-1}\right)+\frac{p_{i}}{2 h}\left(y_{i+1}-y_{i-1}\right)-\frac{h^{2}}{6} p_{i} y_{i}^{\prime \prime \prime}-\frac{h^{2}}{12} y_{i}^{(4)}-\frac{h^{4}}{120} p_{i} y_{i}^{(5)}+q_{i} y_{i}=r_{i}+\tau \tag{8}
\end{equation*}
$$

where, $\tau=\frac{h^{4}}{360} y^{(6)}\left(\xi_{2}\right)-p_{i} \tau_{1}-\tau_{2}$ is the local truncation error and $p\left(x_{i}\right)=p_{i}, q\left(x_{i}\right)=q_{i}$, $r\left(x_{i}\right)=r_{i}$.
By successively differentiating both sides of Equation (4) and evaluating at $x_{i}$, and using into Equation (8), we obtain:

$$
\begin{equation*}
\frac{1}{h^{2}}\left(y_{i+1}-2 y_{i}+y_{i-1}\right)+\frac{p_{i}}{2 h}\left(y_{i+1}-y_{i-1}\right)+A_{i} y_{i}^{\prime \prime}+B_{i} y_{i}^{\prime}+C_{i} y_{i}=H_{i}, \text { for } i=1,2, \mathrm{~L}, N-1 \tag{9}
\end{equation*}
$$

where,

$$
\begin{aligned}
A_{i}= & \frac{h^{2}}{6} p_{i}^{2}-\frac{h^{2}}{12}\left(p_{i}^{2}-2 p_{i}^{\prime}-q_{i}\right)-\frac{h^{4}}{120} p_{i}\left(2 p_{i} p_{i}^{\prime}-3 p_{i}^{\prime \prime}-3 q_{i}^{\prime}+p_{i}\left(p_{i}^{\prime}+q_{i}\right)-p_{i}\left(p_{i}^{2}-2 p_{i}^{\prime}-q_{i}\right)\right) \\
B_{i}= & \frac{h^{2}}{6} p_{i}\left(p_{i}^{\prime}+q_{i}\right)-\frac{h^{2}}{12}\left(p_{i}\left(p_{i}^{\prime}+q_{i}\right)-p_{i}^{\prime \prime}-2 q_{i}^{\prime}\right)-\frac{h^{4}}{120} p_{i}\left\{p_{i}^{\prime}\left(p_{i}^{\prime}+q_{i}\right)+p_{i}\left(p_{i}^{\prime \prime}+q_{i}^{\prime}\right)-p_{i}^{\prime \prime \prime}\right. \\
& \left.-3 q_{i}^{\prime \prime}+p_{i} q_{i}^{\prime}-\left(p_{i}^{\prime}+q_{i}\right)\left(p_{i}^{2}-2 p_{i}^{\prime}-q_{i}\right)\right\} \\
C_{i}= & \frac{h^{2}}{6} p_{i} q_{i}^{\prime}-\frac{h^{2}}{12}\left(p_{i} q_{i}^{\prime}-q_{i}^{\prime \prime}\right)-\frac{h^{4}}{120} p_{i}\left(p_{i}^{\prime} q_{i}^{\prime}+p_{i} q_{i}^{\prime \prime}-q_{i}^{\prime \prime \prime}-q_{i}^{\prime}\left(p_{i}^{2}-2 p_{i}^{\prime}-q_{i}\right)\right)+q_{i} \\
H_{i}= & r_{i}+\left(\frac{h^{2}}{12} p_{i}+\frac{h^{4}}{120} p_{i}\left(p_{i}^{2}-3 p_{i}^{\prime}-q_{i}\right)\right) r_{i}^{\prime}+\left(\frac{h^{2}}{12}-\frac{h^{4}}{120} p_{i}^{2}\right) r_{i}^{\prime \prime}+\frac{h^{4}}{120} p_{i} r_{i}^{\prime \prime \prime}
\end{aligned}
$$

Now, using central difference approximation for $y_{i}^{\prime \prime}$ and $y_{i}^{\prime}$ in Equation (9) and further simplifying, we get:

$$
\begin{equation*}
L^{N} \equiv E_{i} y_{i-1}-F_{i} y_{i}+G_{i} y_{i+1}=H_{i}, \text { for } i=1,2, \ldots, N-1 \tag{10}
\end{equation*}
$$

where,

$$
E_{i}=\frac{1}{h^{2}}-\frac{p_{i}}{2 h}+\frac{A_{i}}{h^{2}}-\frac{B_{i}}{2 h}
$$

$$
F_{i}=\frac{2}{h^{2}}+\frac{2 A_{i}}{h^{2}}-C_{i}
$$

$$
G_{i}=\frac{1}{h^{2}}+\frac{p_{i}}{2 h}+\frac{A_{i}}{h^{2}}+\frac{B_{i}}{2 h}
$$

The tri-diagonal system in Equation (10) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

## 3. Stability and convergence analysis

Case 1: Layer behaviour (i.e. $a(x)+b(x)<0$, for $x \in(0,1)$. Thus $q(x)<0$, since $\varepsilon>0)$

## Lemma 1.

If $y(0) \geq 0$ and $L y(x) \leq 0$, for all $x \in(0,1)$, then the solution $y(x) \geq 0$ for all $x \in(0,1)$ for Equations (4) and (5).

## Proof.

Suppose $t \in(0,1)$, such that $y(t)=\min _{x \in(0,1)} y(x)$ and $y(t)<0$. Since, $t \notin\{0,1\}$ and is a point of minima, then $y^{\prime}(t)=0$ and $y^{\prime \prime}(t) \geq 0$.
Therefore, we have:

$$
L y(t) \equiv y^{\prime \prime}(t)+p(t) y^{\prime}(t)+q(t) y(t)>0,
$$

since $y(t)<0$ (by the assumption) and $q(t)<0$. But, this is a contradiction. Then, it follows that $y(t) \geq 0$ and therefore, $y(x) \geq 0$ for all $x \in(0,1)$.

## Theorem 1.

If the solution of the problem in Equations (4) and (5) satisfies

$$
|y(x)| \leq C \max \left\{|y(0)|, \max _{x \in(0,1)}|L y(x)|\right\}
$$

for some constant $C \geq 1$, then the solution is stable.

## Proof.

We define two functions, $\psi^{ \pm}=C \max \left\{|y(0)|, \max _{x \in(0,1)}|L y(x)|\right\} \pm y(x)$. Then, we get $\psi^{ \pm}(0) \geq 0$ and $L \psi^{ \pm}(x) \equiv C q(x) \max \left\{|y(0)|, \max _{x \in(0,1)}|L y(x)|\right\} \pm L y(x) \leq 0$, since $q(x)<0$ and for suitable choice of $C$. Therefore, by Lemma 1, we get, $\psi^{ \pm}(x) \geq 0$, for all $x \in(0,1)$. So,

$$
|y(x)| \leq C \max \left\{|y(0)|, \max _{x \in(0,1)}|L y(x)|\right\} .
$$

Hence, the stability of the solutions of the problem in Equations (4) and (5) is proved for the case of layer behaviour.

## Lemma 2.

The finite difference operator $L^{N}$ in Equation (10) satisfies the discrete minimum principle. i.e. if $w_{i}$ is any mesh function such that $w_{0} \geq 0$ and $L^{N} w_{i} \leq 0$, for all $x_{i} \in(0,1)$, then $w_{i} \geq 0$ for all $x \in(0,1)$.

## Proof.

Suppose there exists a positive integer $k$ such that $w_{k}<0$ and $w_{k}=\min _{0 \leq i \leq N} w_{i}$. Then, from Equation (10), we have:

$$
\begin{aligned}
L^{N} w_{k} \equiv & E_{k} w_{k-1}-F_{k} w_{k}+G_{k} w_{k+1} \\
& =\left\{\left(\frac{1}{h^{2}}+\frac{p_{k}^{2}}{12}+\frac{p_{k}^{\prime}}{6}+\frac{q_{k}}{12}\right)+\frac{h^{2} p_{k}}{120}\left(3 p_{k}^{\prime \prime}+3 q_{k}^{\prime}-p_{k}\left(p_{k}^{\prime}+q_{k}\right)+p_{k}\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)-2 p_{k} p_{k}^{\prime}\right)\right\}\left(w_{k-1}-w_{k}\right) \\
& +\left\{\left(\frac{1}{h^{2}}+\frac{p_{k}^{2}}{12}+\frac{p_{k}^{\prime}}{6}+\frac{q_{k}}{12}\right)+\frac{h^{2} p_{k}}{120}\left(3 p_{k}^{\prime \prime}+3 q_{k}^{\prime}-p_{k}\left(p_{k}^{\prime}+q_{k}\right)+p_{k}\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)-2 p_{k} p_{k}^{\prime}\right)\right\}\left(w_{k+1}-w_{k}\right) \\
& +\left\{\frac{p_{k}}{2 h}+h\left(\frac{1}{24}\left(p_{k}^{\prime \prime}+2 q_{k}^{\prime}-p_{k} p_{k}^{\prime}-p_{k} q_{k}\right)+\frac{p_{k}}{12}\left(p_{k}^{\prime}+q_{k}\right)\right)\right. \\
& \left.+\frac{h^{3} p_{k}}{240}\left(-p_{k}^{\prime}\left(p_{k}^{\prime}+q_{k}\right)-p_{k}\left(p_{k}^{\prime \prime}+q_{k}^{\prime}\right)+p_{k}^{\prime \prime \prime}+3 q_{k}^{\prime \prime}-p_{k} q_{k}^{\prime}+\left(p_{k}^{\prime}+q_{k}\right)\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)\right)\right\}\left(w_{k+1}-w_{k-1}\right) \\
& +C_{k} w_{k}
\end{aligned}
$$

For sufficiently small $h$ and for suitable value of $p_{k}$, we obtain $L^{N} w_{k}>0$. Since, $w_{k}<0$ (by the assumption) and $C_{k} \rightarrow q_{k}<0$. But, this is a contradiction. Hence, $w_{i} \geq 0$ for all $x_{i} \in(0,1)$.

## Theorem 2.

The finite difference operator $L^{N}$ in Equation (10) is stable for $a(x)+b(x)<0$, if $w_{i}$ is any mesh function, then $\left|w_{i}\right| \leq C \max \left\{\left|w_{0}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\}$, for some constant $C \geq 1$.

## Proof.

We define two functions, $\psi_{i}^{ \pm} \equiv C \max \left\{\left|w_{0}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\} \pm w_{i}$. Then, similar to Theorem 1, we get $\psi_{0}^{ \pm} \geq 0$ and $L \psi_{i}^{ \pm} \equiv C q_{i} \max \left\{\left|w_{0}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\} \pm L w_{i} \leq 0$, since $a_{i}+b_{i}<0 \Rightarrow q_{i}<0$ and $C \geq 1$.
Therefore by Lemma 2 we get

$$
\psi_{i}^{ \pm} \geq 0, \text { for all } x_{i} \in(0,1) . \Rightarrow \psi_{i}^{ \pm} \equiv C \max \left\{\left|w_{o}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\} \pm w_{i} \geq 0 .
$$

Thus, $\left|w_{i}\right| \leq C \max \left\{\left|w_{o}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\}$.
This proves the stability of the scheme for the case of layer behaviour.
Case 2: Oscillatory behaviour (i.e. $a(x)+b(x)>0$, for $x \in(0,1)$. Thus $q(x)>0$, as $\varepsilon>0)$.
The continuous maximum principle and stability of the solution of Equations (4) and (5) are presented as follows for the case of oscillatory behaviour.

## Lemma 3.

If $y(0) \geq 0$ and $L y(x) \geq 0$, for all $x \in(0,1)$, then the solution $y(x) \geq 0$ for all $x \in(0,1)$ for Equations (4) and (5).

## Proof.

Suppose $t \in(0,1)$, such that $y(t)=\max _{x \in(0,1)} y(x)$ and $y(t)<0$. Since, $t \notin\{0,1\}$ and is a point of maxima, therefore $y^{\prime}(t)=0$ and $y^{\prime \prime}(t) \leq 0$. Therefore, we have:

$$
L y(t) \equiv y^{\prime \prime}(t)+p(t) y^{\prime}(t)+q(t) y(t)<0,
$$

since $y(t)<0$ (by the assumption) and $q(t)>0$. But, this is a contradiction. Hence, $y(x) \geq 0$, for all $x \in(0,1)$.

## Theorem 3.

The solution of the problem in Equations (4) and (5) satisfies $|y(x)| \leq K \max \left\{|y(0)|, \max _{x \in(0,1)}|L y(x)|\right\}$, for some constant $K \geq 1$.

## Proof.

The proof is analogous to Theorem 1. Hence, the stability of the solutions of the problem in Equations (4) and (5) is proved for the case of oscillatory behavior. Now, we present the stability of the discrete problem in Equation (10) for the case of oscillatory behavior.

## Lemma 4.

The finite difference operator $L^{N}$ in Equation (10) satisfies the discrete maximum principle, if $w_{i}$ is any mesh function such that $w_{0} \geq 0$ and $L^{N} w_{i} \geq 0$, for all $x_{i} \in(0,1)$, then $w_{i} \geq 0$ for all $x \in(0,1)$.

## Proof.

Suppose that there exists a positive integer $k$ such that $w_{k}<0$ and $w_{k}=\max _{0 \leq i \leq N} w_{i}$. Then, from Equation (10), we have

$$
\begin{aligned}
L^{N} w_{k} & \equiv E_{k} w_{k-1}-F_{k} w_{k}+G_{k} w_{k+1} \\
= & \left\{\left(\frac{1}{h^{2}}+\frac{p_{k}^{2}}{12}+\frac{p_{k}^{\prime}}{6}+\frac{q_{k}}{12}\right)+\frac{h^{2} p_{k}}{120}\left(3 p_{k}^{\prime \prime}+3 q_{k}^{\prime}-p_{k}\left(p_{k}^{\prime}+q_{k}\right)+p_{k}\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)-2 p_{k} p_{k}^{\prime}\right)\right\}\left(w_{k-1}-w_{k}\right) \\
& +\left\{\left(\frac{1}{h^{2}}+\frac{p_{k}^{2}}{12}+\frac{p_{k}^{\prime}}{6}+\frac{q_{k}}{12}\right)+\frac{h^{2} p_{k}}{120}\left(3 p_{k}^{\prime \prime}+3 q_{k}^{\prime}-p_{k}\left(p_{k}^{\prime}+q_{k}\right)+p_{k}\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)-2 p_{k} p_{k}^{\prime}\right)\right\}\left(w_{k+1}-w_{k}\right) \\
& +\left\{\frac{p_{k}}{2 h}+h\left(\frac{1}{24}\left(p_{k}^{\prime \prime}+2 q_{k}^{\prime}-p_{k} p_{k}^{\prime}-p_{k} q_{k}\right)+\frac{p_{k}}{12}\left(p_{k}^{\prime}+q_{k}\right)\right)\right. \\
& \left.+\frac{h^{3} p_{k}}{240}\left(-p_{k}^{\prime}\left(p_{k}^{\prime}+q_{k}\right)-p_{k}\left(p_{k}^{\prime \prime}+q_{k}^{\prime}\right)+p_{k}^{\prime \prime \prime}+3 q_{k}^{\prime \prime}-p_{k} q_{k}^{\prime}+\left(p_{k}^{\prime}+q_{k}\right)\left(p_{k}^{2}-2 p_{k}^{\prime}-q_{k}\right)\right)\right\}\left(w_{k+1}-w_{k-1}\right) \\
& +C_{k} w_{k}
\end{aligned}
$$

For sufficiently small $h$ and for suitable value of $p_{k}$, we obtain $L^{N} w_{k}<0$. Since, $w_{k}<0$ (by the assumption) and $C_{k} \rightarrow q_{k}>0$. But, this is a contradiction. Hence, $w_{i} \geq 0$ for all $x_{i} \in(0,1)$.

## Theorem 4.

The finite difference operator $L^{N}$ in Equation (10) is stable for $a(x)+b(x)>0,($ i.e. $q(x)>0)$, if $w_{i}$ is any mesh function, then $\left|w_{i}\right| \leq K \max \left\{\left|w_{0}\right|, \max _{x_{i} \in(0,1)}\left|L w_{i}\right|\right\}$, for some constant $K \geq 1$.

## Proof.

The proof is similar to Theorem 2. This proves the stability of the scheme for the case of oscillatory behaviour.

## Definition (Uniform convergence)

Let $y$ be a solution of Equations (1) and (2). Consider a difference scheme for solving Equations (1) and (2). If the scheme has a numerical solution $y^{N}$ that satisfies $\left\|y-y^{N}\right\| \leq C h^{p}$, where $C>0$ and $p>0$ are independent of $\varepsilon$ and of the mesh size $h$, then we say the scheme uniformly converges to $y$ with respect to the norm ||.\| [14].

## Theorem 5.

Let $y(x)$ be the analytical solution of the problem in Equation (4) and Equation (5) and $y^{N}(x)$ be the numerical solution of the discretized problem of Equation (10). Then, $\left\|y-y^{N}\right\| \leq C^{*} h^{4}$ for sufficiently small $h$ and $C^{*}$ is positive constant.

## Proof.

Multiplying both sides of Equation (10) by $-h^{2}$, we obtain

$$
\begin{equation*}
\left(-1+u_{i}\right) y_{i-1}+\left(2+v_{i}\right) y_{i}+\left(-1+w_{i}\right) y_{i+1}+g_{i}+\tau_{i}(h)=0 \tag{11}
\end{equation*}
$$

where, $\tau_{i}(h)=\frac{h^{6}}{360} y^{(6)}\left(\xi_{2}\right)+O\left(h^{8}\right)$ is a local truncation error, for $i=1,2, \ldots, N-1$.

$$
\begin{aligned}
& u_{i}=\frac{h}{2} p_{i}-A_{i}+\frac{h}{2} B_{i} \\
& v_{i}=2 A_{i}-h^{2} C_{i} \\
& w_{i}=-\frac{h}{2} p_{i}-A_{i}-\frac{h}{2} B_{i} \\
& g_{i}=h^{2} H_{i}
\end{aligned}
$$

Incorporating the boundary conditions $y_{0}=\phi\left(x_{0}\right)=\phi_{0}, y_{N}=y(1)=\beta$ in Equation (11), we get the systems of equations of the form:

$$
\begin{equation*}
(D+R) y+Z+\tau(h)=0 \tag{12}
\end{equation*}
$$

where,

$$
D=\left[\begin{array}{ccccc}
2 & -1 & 0 & \mathrm{~L} & 0 \\
-1 & 2 & -1 & \mathrm{~L} & 0 \\
0 & - & - & & - \\
\mathrm{M} & & & \mathrm{O} & \mathrm{M} \\
0 & - & - & -1 & 2
\end{array}\right], \quad R=\left[\begin{array}{ccccc}
v_{1} & w_{1} & 0 & \mathrm{~L} & 0 \\
u_{2} & v_{2} & w_{2} & \mathrm{~L} & 0 \\
0 & - & - & & - \\
\mathrm{M} & & \mathrm{O} & \mathrm{M} \\
0 & - & - & u_{N-1} & v_{N-1}
\end{array}\right] \text { are tri-diagonal matrices of }
$$

order $N-1$, and
$Z=\left[\left(g_{1}+\left(-1+u_{1}\right) \phi(0)\right), g_{2}, g_{3}, \mathrm{~L},\left(g_{N-1}+\left(-1+w_{N-1}\right) \beta\right)\right]^{T}, \quad \tau(h)=O\left(h^{4}\right)$ and
$y=\left[y_{1}, y_{2}, \mathrm{~L}, y_{N-1}\right]^{T}, \tau(h)=\left[\tau_{1}, \tau_{2}, \mathrm{~L}, \tau_{N-1}\right]^{T}, \overline{0}=[0,0, \mathrm{~L}, 0]^{T}$
are the associated vectors of Equation (12).
Let $y^{N}=\left[y_{1}^{N}, y_{2}^{N}, \mathrm{~L}, y_{N-1}^{N}\right]^{T} \cong y$ be the solution which satisfies the Equation (12), then we have:

$$
\begin{equation*}
(D+R) y^{N}+Z=0 \tag{13}
\end{equation*}
$$

Let $e_{i}=y_{i}-y_{i}^{N}$, for $i=1,2, \mathrm{~L}, N-1$ be the discretization error, then, $y-y^{N}=\left[e_{1}, e_{2}, \mathrm{~L}, e_{N-1}\right]^{T}$.
Subtracting Equation (12) from Equation (13), we get

$$
\begin{equation*}
(D+R)\left(y^{N}-y\right)=\tau(h) \tag{14}
\end{equation*}
$$

Let $\left|p_{i}\right| \leq M_{1},\left|p_{i}^{\prime}\right| \leq M_{2},\left|p_{i}^{\prime \prime}\right| \leq M_{3},\left|q_{i}\right| \leq K_{1},\left|q_{i}^{\prime}\right| \leq K_{2},\left|q_{i}^{\prime \prime}\right| \leq K_{3}$
Let $r_{i j}$ be the $(i, j)^{t h}$ element of the matrix $R$, then for $i=1,2, \mathrm{~L}, N-2$

$$
\begin{aligned}
\left|r_{i, i+1}\right|= & \left|w_{i}\right| \leq h\left\{\frac{M_{1}}{2}+\frac{h}{6} M_{1}^{2}+\frac{h}{12}\left(M_{1}^{2}-2 M_{2}-K_{1}\right)\right. \\
& \left.+\frac{h^{3} M_{1}}{120}\left(2 M_{1} M_{2}-3 M_{3}-3 K_{2}+M_{1}\left(M_{2}+K_{1}\right)-M_{1}\left(M_{1}^{2}-2 M_{2}-K_{1}\right)\right)+\frac{B_{i}}{2}\right\}
\end{aligned}
$$

For $i=2,3, \mathrm{~L}, N-1$

$$
\begin{aligned}
\left|r_{i, i-1}\right|= & \left|u_{i}\right| \leq h\left\{\frac{M_{1}}{2}+\frac{h}{6} M_{1}^{2}+\frac{h}{12}\left(M_{1}^{2}-2 M_{2}-K_{1}\right)\right. \\
& \left.+\frac{h^{3} M_{1}}{120}\left(2 M_{1} M_{2}-3 M_{3}-3 K_{2}+M_{1}\left(M_{2}+K_{1}\right)-M_{1}\left(M_{1}^{2}-2 M_{2}-K_{1}\right)\right)+\frac{B_{i}}{2}\right\}
\end{aligned}
$$

Thus, for sufficiently small $h$, we have

$$
\begin{aligned}
& -1+\left|r_{i, i+1}\right|<0, \quad i=1,2, \mathrm{~L}, N-2 \\
& -1+\left|r_{i, i-1}\right|<0, \quad i=2,3, \mathrm{~L}, N-1 .
\end{aligned}
$$

Hence, the matrix $(D+R)$ is irreducible [15].
Let $S_{i}$ be the sum of the elements of the $i^{\text {th }}$ row of the matrix $(D+R)$, then;

$$
\begin{aligned}
S_{i}=1+v_{i}+w_{i}= & 1+h\left(\frac{-p_{i}}{2}\right)+h^{2}\left(\frac{p_{i}^{2}}{12}+\frac{p_{i}^{\prime}}{6}-\frac{11}{12} q_{i}\right) \\
& +h^{3}\left(\frac{1}{24}\left(p_{i} p_{i}^{\prime}+p_{i} q_{i}-p_{i}^{\prime \prime}-2 q_{i}^{\prime}\right)-\frac{1}{12} p_{i}\left(p_{i}^{\prime}+q_{i}\right)\right)+O\left(h^{4}\right), \text { for } i=1 \\
S_{i}=u_{i}+v_{i}+w_{i} & =h^{2}\left(-q_{i}\right)+O\left(h^{4}\right), \text { for } i=2,3, \mathrm{~L}, N-2 \\
S_{i}=1+u_{i}+v_{i}= & 1+h\left(\frac{p_{i}}{2}\right)+h^{2}\left(\frac{p_{i}^{2}}{12}+\frac{p_{i}^{\prime}}{6}-\frac{11}{12} q_{i}\right) \\
& +h^{3}\left(\frac{1}{24}\left(-p_{i} p_{i}^{\prime}-p_{i} q_{i}+p_{i}^{\prime \prime}+2 q_{i}^{\prime}\right)+\frac{1}{12} p_{i}\left(p_{i}^{\prime}+q_{i}\right)\right)+O\left(h^{4}\right), \text { for } i=N-1
\end{aligned}
$$

Let $M_{1^{*}}=\min _{1 \leq i \leq N-1}\left|p_{i}\right|, M_{1}^{*}=\max _{1 \leq i \leq N-1}\left|p_{i}\right|, K_{1^{*}}=\min _{1 \leq i \leq N-1}\left|q_{i}\right|, K_{1}^{*}=\max _{1 \leq i \leq N-1}\left|q_{i}\right|$, then:

$$
0<M_{1^{*}} \leq M_{1} \leq M_{1}^{*} \text { and } 0<K_{1^{*}} \leq K_{1} \leq K_{1}^{*}
$$

For sufficiently small $h,(D+R)$ is monotone [15-16].
Hence, $(D+R)^{-1}$ exists and $(D+R)^{-1} \geq 0$.
From the error Equation (14), we have:

$$
\begin{equation*}
\left\|y-y^{N}\right\| \leq\left\|(D+R)^{-1}\right\|\|\tau(h)\| \tag{15}
\end{equation*}
$$

For sufficiently small $h$, we have:
$S_{i}>\frac{11}{12} h^{2} \mathrm{~K}_{\mathrm{1}^{*}}$, for $i=1$
$S_{i}>h^{2} \mathrm{~K}_{1^{*}}$, for $i=2,3, \mathrm{~L}, N-2$
$S_{i}>\frac{11}{12} h^{2} \mathrm{~K}_{1^{*}}$, for $i=N-1, \quad$ where, $K_{1^{*}}=\min _{1 \leq i \leq N-1}\left|q_{i}\right|$
Let $(D+R)_{i, k}^{-1}$ be the $(i, k)^{t h}$ element of $(D+R)^{-1}$ and we define,
$\left\|(D+R)^{-1}\right\|=\max _{1 \leq i \leq N-1} \sum_{k=1}^{N-1}(D+R)_{i, k}^{-1}$ and $\|\tau(h)\|=\max _{1 \leq i \leq N-1}\left|\tau_{i}\right|$
Since $(D+R)_{i, k}^{-1} \geq 0$, then from the theory of matrices, we have

$$
\begin{equation*}
\sum_{k=1}^{N-1}(D+R)_{i, k}^{-1} \cdot S_{k}=1, \quad i=1,2, \mathrm{~L}, N-1 . \tag{17}
\end{equation*}
$$

Hence, $(D+R)_{i, 1}^{-1} \leq \frac{1}{S_{1}}<\frac{12}{11 h^{2} Q^{*}}$, for $k=1$, since $0<\varepsilon \ll 1$.
$(D+R)_{i, N-1}^{-1} \leq \frac{1}{S_{N-1}}<\frac{12}{11 h^{2} Q^{*}}$, for $k=N-1$
Further, $\sum_{k=2}^{N-2}(D+R)_{i, k}^{-1} \leq \frac{1}{\min _{2 \leq k \leq N-2} S_{k}}<\frac{1}{h^{2} Q^{*}}$, for $k=2,3, \mathrm{~L}, N-2$
where, $Q^{*}=\min _{1 \leq i \leq N-1}\left|a_{i}+b_{i}\right|$, since $q\left(x_{i}\right)=\left(\frac{a\left(x_{i}\right)+b\left(x_{i}\right)}{\varepsilon}\right)$.
Now, from Equations (15) - (19), we get

$$
\begin{equation*}
\left\|y-y^{N}\right\| \leq \frac{7}{792}\left(\frac{y^{(6)}\left(\xi_{2}\right)}{Q^{*}}\right) h^{4}=C^{*} h^{4} \tag{20}
\end{equation*}
$$

where, $C^{*}=\frac{7}{792}\left(\frac{y^{(6)}\left(\xi_{2}\right)}{Q^{*}}\right)$, which is independent of perturbation parameter $\varepsilon$ and mesh size
$h$. This establishes that the present method is of fourth order uniformly convergent.

## 4. Numerical examples and results

To demonstrate the applicability of the method, we implemented the method on four numerical examples, two with twin boundary layers and two with oscillatory behaviour. Since, those examples have no exact solution, so the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double-mesh principle given by

$$
\begin{equation*}
Z_{h}=\max _{i}\left|y_{i}^{h}-y_{i}^{h / 2}\right|, \quad i=1,2, \ldots, N-1 \tag{21}
\end{equation*}
$$

where $y_{i}^{h}$ is the numerical solution on the mesh $\left\{x_{i}\right\}_{1}^{N-1}$ at the nodal point $x_{i}$ and $x_{i}=x_{0}+i h$, $i=1,2, \ldots, N-1$ and $y_{i}^{h / 2}$ is the numerical solution on a mesh, obtained by bisecting the original mesh with $N$ number of mesh intervals [4].

## Example 1.

Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour,

$$
\varepsilon y^{\prime \prime}(x)+0.25 y(x-\delta)-y(x)=1
$$

under the interval and boundary conditions

$$
y(x)=1,-\delta \leq x \leq 0 \text { and } y(1)=0 .
$$

The maximum absolute errors are presented in Tables 1 and 2 for different values of $\varepsilon$ and $\delta$.
The graph of the computed solution for $\varepsilon=0.01$ and different values of $\delta$ is also given in Figure 1.

## Example 2.

Consider the singularly perturbed delay reaction-diffusion equation with layer behaviour,

$$
\varepsilon y^{\prime \prime}(x)-2 y(x-\delta)-y(x)=1
$$

under the interval and boundary conditions

$$
y(x)=1,-\delta \leq x \leq 0 \text { and } y(1)=0 .
$$

The maximum absolute errors are presented in Tables 3 and 4 for different values of $\varepsilon$ and $\delta$. The graph of the computed solution for $\varepsilon=0.01$ and different values of $\delta$ is also given in Figure 2.

Table 1. The maximum absolute errors of Example 1, for different values of $\delta$ with $\varepsilon=0.1$.

| $\delta \downarrow$ | $N=100$ | $N=200$ | $N=300$ | $N=400$ | $N=500$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Our Method |  |  |  |  |  |
| 0.03 | $1.2007 \mathrm{e}-09$ | $7.5051 \mathrm{e}-10$ | $1.4815 \mathrm{e}-11$ | $4.6660 \mathrm{e}-12$ | $1.9661 \mathrm{e}-12$ |
| 0.05 | $1.2135 \mathrm{e}-09$ | $7.5860 \mathrm{e}-10$ | $1.4980 \mathrm{e}-11$ | $4.7337 \mathrm{e}-12$ | $1.9339 \mathrm{e}-12$ |
| 0.09 | $1.2290 \mathrm{e}-09$ | $7.6818 \mathrm{e}-10$ | $1.5168 \mathrm{e}-11$ | $4.7632 \mathrm{e}-12$ | $2.0450 \mathrm{e}-12$ |
| Results in $[10]$ |  |  |  |  |  |
| 0.03 | $2.1999 \mathrm{e}-03$ | $1.1041 \mathrm{e}-03$ | $7.3705 \mathrm{e}-04$ | $5.5315 \mathrm{e}-04$ | $4.4269 \mathrm{e}-04$ |
| 0.05 | $2.2012 \mathrm{e}-03$ | $1.1049 \mathrm{e}-03$ | $7.3749 \mathrm{e}-04$ | $5.5345 \mathrm{e}-04$ | $4.4293 \mathrm{e}-04$ |
| 0.09 | $2.1999 \mathrm{e}-03$ | $1.1038 \mathrm{e}-03$ | $7.3676 \mathrm{e}-04$ | $5.5289 \mathrm{e}-04$ | $4.4247 \mathrm{e}-04$ |

Table 2. The maximum absolute errors of Example 1, for different values of $\varepsilon$ with $\delta=0.5 \varepsilon$.

| $\varepsilon \downarrow$ | $N=2^{4}$ | $N=2^{5}$ | $N=2^{6}$ | $N=2^{7}$ | $N=2^{8}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Our Method |  |  |  |  |  |
| $2^{-4}$ | $4.7163 \mathrm{e}-06$ | $2.9533 \mathrm{e}-07$ | $1.8473 \mathrm{e}-08$ | $1.1546 \mathrm{e}-09$ | $7.2184 \mathrm{e}-11$ |
| $2^{-5}$ | $1.6851 \mathrm{e}-05$ | $1.0582 \mathrm{e}-06$ | $6.6233 \mathrm{e}-08$ | $4.1407 \mathrm{e}-09$ | $2.5883 \mathrm{e}-10$ |
| $2^{-6}$ | $6.1305 \mathrm{e}-05$ | $3.9010 \mathrm{e}-06$ | $2.4413 \mathrm{e}-07$ | $1.5281 \mathrm{e}-08$ | $9.5513 \mathrm{e}-10$ |
| $2^{-7}$ | $2.3541 \mathrm{e}-04$ | $1.5098 \mathrm{e}-05$ | $9.4835 \mathrm{e}-07$ | $5.9419 \mathrm{e}-08$ | $3.7143 \mathrm{e}-09$ |
| $2^{-8}$ | $9.2982 \mathrm{e}-04$ | $5.9195 \mathrm{e}-05$ | $3.7478 \mathrm{e}-06$ | $2.3512 \mathrm{e}-07$ | $1.4703 \mathrm{e}-08$ |
| $2^{-9}$ | $3.5840 \mathrm{e}-03$ | $2.3115 \mathrm{e}-04$ | $1.4856 \mathrm{e}-05$ | $9.3248 \mathrm{e}-07$ | $5.8449 \mathrm{e}-08$ |
| $2^{-10}$ | $1.1856 \mathrm{e}-02$ | $9.1935 \mathrm{e}-04$ | $5.8565 \mathrm{e}-05$ | $3.7066 \mathrm{e}-06$ | $2.3261 \mathrm{e}-07$ |
| Results in $[10]$ |  |  |  |  |  |
| $2^{-4}$ | $1.8632 \mathrm{e}-02$ | $9.6189 \mathrm{e}-03$ | $4.8865 \mathrm{e}-03$ | $2.4643 \mathrm{e}-03$ | $1.2376 \mathrm{e}-03$ |
| $2^{-5}$ | $2.8161 \mathrm{e}-02$ | $1.4818 \mathrm{e}-02$ | $7.6255 \mathrm{e}-03$ | $3.8713 \mathrm{e}-03$ | $1.9509 \mathrm{e}-03$ |
| $2^{-6}$ | $3.7958 \mathrm{e}-02$ | $2.0967 \mathrm{e}-02$ | $1.0977 \mathrm{e}-02$ | $5.6273 \mathrm{e}-03$ | $2.8498 \mathrm{e}-03$ |
| $2^{-7}$ | $5.0640 \mathrm{e}-02$ | $2.8316 \mathrm{e}-02$ | $1.5267 \mathrm{e}-02$ | $7.9105 \mathrm{e}-03$ | $4.0287 \mathrm{e}-03$ |
| $2^{-8}$ | $6.3580 \mathrm{e}-02$ | $3.7706 \mathrm{e}-02$ | $2.0984 \mathrm{e}-02$ | $1.1012 \mathrm{e}-02$ | $5.6555 \mathrm{e}-03$ |
| $2^{-9}$ | $8.3843 \mathrm{e}-02$ | $5.0477 \mathrm{e}-02$ | $2.8297 \mathrm{e}-02$ | $1.5261 \mathrm{e}-02$ | $7.9111 \mathrm{e}-03$ |
| $2^{-10}$ | $9.9137 \mathrm{e}-02$ | $6.3529 \mathrm{e}-02$ | $3.7660 \mathrm{e}-02$ | $2.0974 \mathrm{e}-02$ | $1.1011 \mathrm{e}-02$ |



Figure 1. The numerical solution of Example 1 with $\varepsilon=0.01$ and $N=100$.

Table 3. The maximum absolute errors of Example 2, for different values of $\delta$ with $\varepsilon=0.1$.

| $\delta \downarrow$ |  | $N=100$ | $N=200$ | $N=300$ | $N=400$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Our Method |  |  |  | $N=500$ |  |
| 0.03 | $5.9892 \mathrm{e}-09$ | $3.7452 \mathrm{e}-10$ | $7.3976 \mathrm{e}-11$ | $2.3404 \mathrm{e}-11$ | $9.5863 \mathrm{e}-12$ |
| 0.05 | $3.3028 \mathrm{e}-09$ | $2.0657 \mathrm{e}-10$ | $4.0807 \mathrm{e}-11$ | $1.2909 \mathrm{e}-11$ | $5.2809 \mathrm{e}-12$ |
| 0.09 | $4.6352 \mathrm{e}-09$ | $2.8949 \mathrm{e}-10$ | $5.7180 \mathrm{e}-11$ | $1.8085 \mathrm{e}-11$ | $7.4190 \mathrm{e}-12$ |
| Results in $[10]$ |  |  |  |  |  |
| 0.03 | $3.1674 \mathrm{e}-03$ | $1.6058 \mathrm{e}-03$ | $1.0754 \mathrm{e}-03$ | $8.0837 \mathrm{e}-04$ | $6.4760 \mathrm{e}-04$ |
| 0.05 | $3.1437 \mathrm{e}-03$ | $1.5949 \mathrm{e}-03$ | $1.0685 \mathrm{e}-03$ | $8.0338 \mathrm{e}-04$ | $6.4367 \mathrm{e}-04$ |
| 0.09 | $3.0784 \mathrm{e}-03$ | $1.5660 \mathrm{e}-03$ | $1.0502 \mathrm{e}-03$ | $7.9000 \mathrm{e}-04$ | $6.3310 \mathrm{e}-04$ |

Table 4. The maximum absolute errors of Example 2, for different values of $\varepsilon$ with $\delta=0.5 \varepsilon$.

| $\varepsilon \downarrow$ | $N=2^{4}$ | $N=2^{5}$ | $N=2^{6}$ | $N=2^{7}$ | $N=2^{8}$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| Our Method | $1.7218 \mathrm{e}-05$ | $1.0980 \mathrm{e}-06$ | $6.9308 \mathrm{e}-08$ | $4.3372 \mathrm{e}-09$ | $2.7116 \mathrm{e}-10$ |
| $2^{-4}$ | $8.6267 \mathrm{e}-05$ | $5.7179 \mathrm{e}-06$ | $3.5965 \mathrm{e}-07$ | $2.2514 \mathrm{e}-08$ | $1.4086 \mathrm{e}-09$ |
| $2^{-5}$ | $4.0309 \mathrm{e}-04$ | $2.6120 \mathrm{e}-05$ | $1.6483 \mathrm{e}-06$ | $1.0385 \mathrm{e}-07$ | $6.4944 \mathrm{e}-09$ |
| $2^{-6}$ | $1.6675 \mathrm{e}-03$ | $1.1001 \mathrm{e}-04$ | $7.1717 \mathrm{e}-06$ | $4.5007 \mathrm{e}-07$ | $2.8201 \mathrm{e}-08$ |
| $2^{-7}$ | $5.7218 \mathrm{e}-03$ | $4.6571 \mathrm{e}-04$ | $2.9880 \mathrm{e}-05$ | $1.8861 \mathrm{e}-06$ | $1.1867 \mathrm{e}-07$ |
| $2^{-8}$ | $1.5760 \mathrm{e}-02$ | $1.8472 \mathrm{e}-03$ | $1.2042 \mathrm{e}-04$ | $7.7976 \mathrm{e}-06$ | $4.8901 \mathrm{e}-07$ |
| $2^{-9}$ | $3.3872 \mathrm{e}-02$ | $6.2077 \mathrm{e}-03$ | $4.9356 \mathrm{e}-04$ | $3.1554 \mathrm{e}-05$ | $1.9940 \mathrm{e}-06$ |
| $2^{-10}$ |  |  |  |  |  |
| Results in $[10]^{2.1118 \mathrm{e}-02}$ | $1.1692 \mathrm{e}-02$ | $6.1941 \mathrm{e}-03$ | $3.1887 \mathrm{e}-03$ | $1.6178 \mathrm{e}-03$ |  |
| $2^{-4}$ | $2.7872 \mathrm{e}-02$ | $1.6023 \mathrm{e}-02$ | $8.6367 \mathrm{e}-03$ | $4.4957 \mathrm{e}-03$ | $2.2948 \mathrm{e}-03$ |
| $2^{-5}$ | $3.5711 \mathrm{e}-02$ | $2.1293 \mathrm{e}-02$ | $1.1869 \mathrm{e}-02$ | $6.2731 \mathrm{e}-03$ | $3.2240 \mathrm{e}-03$ |
| $2^{-6}$ | $4.6679 \mathrm{e}-02$ | $2.8350 \mathrm{e}-02$ | $1.6107 \mathrm{e}-02$ | $8.6728 \mathrm{e}-03$ | $4.5120 \mathrm{e}-03$ |
| $2^{-7}$ | $5.4895 \mathrm{e}-02$ | $3.6018 \mathrm{e}-02$ | $2.1373 \mathrm{e}-02$ | $1.1929 \mathrm{e}-02$ | $6.2847 \mathrm{e}-03$ |
| $2^{-8}$ | $5.7371 \mathrm{e}-02$ | $4.7254 \mathrm{e}-02$ | $2.8581 \mathrm{e}-02$ | $1.6140 \mathrm{e}-02$ | $8.6961 \mathrm{e}-03$ |
| $2^{-9}$ | $5.7878 \mathrm{e}-02$ | $5.5695 \mathrm{e}-02$ | $3.6153 \mathrm{e}-02$ | $2.1406 \mathrm{e}-02$ | $1.1956 \mathrm{e}-02$ |
| $2^{-10}$ |  |  |  |  |  |



Figure 2. The numerical solution of Example 2 with $\varepsilon=0.01$ and $N=100$.

## Example 3.

Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour,

$$
\varepsilon y^{\prime \prime}(x)+0.25 y(x-\delta)+y(x)=1
$$

under the interval and boundary conditions

$$
y(x)=1,-\delta \leq x \leq 0 \text { and } y(1)=0 .
$$

The maximum absolute errors are presented in Table 5 for different values of $\delta$. The graph of the computed solution for $\varepsilon=0.001$ and different values of $\delta$ is also given in Figure 3.

Table 5. The maximum absolute errors of Example 3, for different values of $\delta$ with $\varepsilon=0.1$.

| $\delta \downarrow$ | $N=100$ | $N=200$ | $N=300$ | $N=400$ | $N=500$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Our Method |  |  |  |  |  |
| 0.03 | $3.9856 \mathrm{e}-08$ | $2.4916 \mathrm{e}-09$ | $4.9143 \mathrm{e}-10$ | $1.5603 \mathrm{e}-10$ | $6.1932 \mathrm{e}-11$ |
| 0.05 | $3.8949 \mathrm{e}-08$ | $2.4343 \mathrm{e}-09$ | $4.8003 \mathrm{e}-10$ | $1.5358 \mathrm{e}-10$ | $7.0907 \mathrm{e}-11$ |
| 0.09 | $3.7554 \mathrm{e}-08$ | $2.3446 \mathrm{e}-09$ | $4.6287 \mathrm{e}-10$ | $1.6033 \mathrm{e}-10$ | $6.1303 \mathrm{e}-11$ |
| Results in $[10]$ |  |  |  |  |  |
| 0.03 | $2.5991 \mathrm{e}-03$ | $1.2872 \mathrm{e}-03$ | $8.5528 \mathrm{e}-04$ | $6.4039 \mathrm{e}-04$ | $5.1179 \mathrm{e}-04$ |
| 0.05 | $2.6270 \mathrm{e}-03$ | $1.3013 \mathrm{e}-03$ | $8.6474 \mathrm{e}-04$ | $6.4750 \mathrm{e}-04$ | $5.1749 \mathrm{e}-04$ |
| 0.09 | $2.6813 \mathrm{e}-03$ | $1.3289 \mathrm{e}-03$ | $8.8320 \mathrm{e}-04$ | $6.6139 \mathrm{e}-04$ | $5.2863 \mathrm{e}-04$ |



Figure 3. The numerical solution of Example 3 with $\varepsilon=0.001$ and $N=100$.

## Example 4.

Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behaviour,

$$
\varepsilon y^{\prime \prime}(x)+y(x-\delta)+2 y(x)=1
$$

under the interval and boundary conditions

$$
y(x)=1,-\delta \leq x \leq 0 \text { and } y(1)=0 .
$$

The maximum absolute errors are presented in Table 6 for different values of $\delta$. The graph of the computed solution for $\varepsilon=0.001$ and different values of $\delta$ is also given in Figure 4.

Table 6. The maximum absolute errors of Example 4, for different values of $\delta$ with $\varepsilon=0.1$.

| $\delta \downarrow$ | $N=100$ | $N=200$ | $N=300$ | $N=400$ | $N=500$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Our Method |  |  |  |  |  |
| 0.03 | $1.5497 \mathrm{e}-07$ | $9.6846 \mathrm{e}-09$ | $1.9131 \mathrm{e}-09$ | $6.0394 \mathrm{e}-10$ | $2.4770 \mathrm{e}-10$ |
| 0.05 | $1.5900 \mathrm{e}-07$ | $9.9375 \mathrm{e}-09$ | $1.9630 \mathrm{e}-09$ | $6.2120 \mathrm{e}-10$ | $2.5444 \mathrm{e}-10$ |
| 0.09 | $1.7208 \mathrm{e}-07$ | $1.0754 \mathrm{e}-08$ | $2.1244 \mathrm{e}-09$ | $6.7226 \mathrm{e}-10$ | $2.7451 \mathrm{e}-10$ |
| Results in $[10]$ |  |  |  |  |  |
| 0.03 | $1.5929 \mathrm{e}-02$ | $7.4850 \mathrm{e}-03$ | $4.8816 \mathrm{e}-03$ | $3.6202 \mathrm{e}-03$ | $2.8764 \mathrm{e}-03$ |
| 0.05 | $1.5470 \mathrm{e}-02$ | $7.2782 \mathrm{e}-03$ | $4.7473 \mathrm{e}-03$ | $3.5209 \mathrm{e}-03$ | $2.7975 \mathrm{e}-03$ |
| 0.09 | $2.1396 \mathrm{e}-02$ | $1.0097 \mathrm{e}-02$ | $6.5922 \mathrm{e}-03$ | $4.8916 \mathrm{e}-03$ | $3.8879 \mathrm{e}-03$ |



Figure 4. The numerical solution of Example 4 with $\varepsilon=0.001$ and $N=100$.

## Illustration of the effect of delay on the solution

The above graphs (Figures 1-4) show the numerical solutions obtained by the present method for different values of delay parameter $\delta$.

## The rate of convergence ( $\rho$ )

In the same way in Equation (21) one can define $Z_{h / 2}$ by replacing $h$ by $h / 2$ and $N-1$ by $2 N-1$, that is:

$$
Z_{h / 2}=\max _{i}\left|y_{i}^{h / 2}-y_{i}^{h / 4}\right| \text {, for } i=1,2, \ldots, 2 N-1 .
$$

The computational rate of convergence $\rho$ is also obtained by using the double mesh principle defined as in [4].

$$
\rho=\frac{\left(\log \left(Z_{h}\right)-\log \left(Z_{h / 2}\right)\right)}{\log 2} .
$$

The following tables (i.e., Tables 7 and 8 ) shows the rate of convergence $\rho$ of the present method for different values of the mesh size $h$.

Table 7. Rate of Convergence $\rho$ for $\varepsilon=0.1$ and $\delta=0.05$.

|  | $h$ | $h / 2$ | $Z_{h}$ | $h / 4$ | $Z_{h / 2}$ | $\rho$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Example 1 |  |  |  |  |  |  |
|  | $1 / 100$ | $1 / 200$ | $1.2135 \mathrm{e}-09$ | $1 / 400$ | $7.5855 \mathrm{e}-11$ | 3.9998 |
|  | $1 / 200$ | $1 / 400$ | $7.5860 \mathrm{e}-11$ | $1 / 800$ | $4.9848 \mathrm{e}-12$ | 3.9277 |
| Example 2 | $1 / 300$ | $1 / 600$ | $1.4980 \mathrm{e}-11$ | $1 / 1200$ | $9.4408 \mathrm{e}-13$ | 3.9880 |
|  |  |  |  |  |  |  |
|  | $1 / 100$ | $1 / 200$ | $3.3028 \mathrm{e}-09$ | $1 / 400$ | $2.0653 \mathrm{e}-10$ | 3.9993 |
|  | $1 / 200$ | $1 / 400$ | $2.0657 \mathrm{e}-10$ | $1 / 800$ | $1.2909 \mathrm{e}-11$ | 4.0002 |
|  | $1 / 300$ | $1 / 600$ | $4.0807 \mathrm{e}-11$ | $1 / 1200$ | $2.5474 \mathrm{e}-12$ | 4.0017 |

Table 8. Rate of Convergence $\rho$ for $\varepsilon=0.1$ and $\delta=0.03$.

|  | $h$ | $h / 2$ | $Z_{h}$ | $h / 4$ | $Z_{h / 2}$ | $\rho$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Example 3 |  |  |  |  |  |  |
|  | $1 / 100$ | $1 / 200$ | $3.9856 \mathrm{e}-08$ | $1 / 400$ | $2.4913 \mathrm{e}-09$ | 3.9998 |
|  | $1 / 200$ | $1 / 400$ | $2.4916 \mathrm{e}-09$ | $1 / 800$ | $1.5603 \mathrm{e}-10$ | 3.9971 |
| Example 4 | $1 / 300$ | $1 / 600$ | $4.9143 \mathrm{e}-10$ | $1 / 1200$ | $3.0611 \mathrm{e}-11$ | 4.0049 |
|  |  |  |  |  |  |  |
|  | $1 / 100$ | $1 / 200$ | $1.5497 \mathrm{e}-07$ | $1 / 400$ | $9.6846 \mathrm{e}-09$ | 4.0001 |
|  | $1 / 200$ | $1 / 400$ | $9.6846 \mathrm{e}-09$ | $1 / 800$ | $6.0394 \mathrm{e}-10$ | 4.0032 |
|  | $1 / 300$ | $1 / 600$ | $1.9131 \mathrm{e}-09$ | $1 / 1200$ | $1.1975 \mathrm{e}-10$ | 3.9978 |

## 5. Discussion and conclusion

Fourth order numerical method for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour has been presented. To demonstrate the efficiency of the method, four model examples without exact solutions have been considered for different values of the perturbation parameter $\varepsilon$ and delay parameter $\delta$. The numerical solutions are tabulated (Tables 1 to 6 ) in terms of maximum absolute errors and observed that the present method improves the findings of Swamy et al. [10]. Also, it is significant that all of the maximum absolute errors decrease rapidly as $N$ increases. The stability and $\varepsilon$-uniform convergence of the method are investigated and established well. The results presented in Tables 7 and 8 confirmed that computational rate of convergence as well as theoretical estimates indicate that method is a fourth order convergent.

Further, to investigate the effect of delay on the solution of the problem, numerical solutions have been presented using graphs. Accordingly, when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour (Figure 1). When the delay parameter is of $O(\varepsilon)$, the solution maintains layer behaviour although the coefficient of the delay term in the equation is of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases (Figure 2). For the oscillatory behaviour case, one can conclude that the solution oscillates throughout the domain for different values of delay parameter $\delta$ (Figures 3 and 4). In a concise manner, the present method gives more accurate solution and is uniformly convergent for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behaviour.

## Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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